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# THE DESIGN OF LOW ORDER CONTROLLERS USING THE FROBENIUS-HANKEL NORM


**Russell Allen Ramaker**

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The problem of determining robust, low order controllers which achieve stability and disturbance attenuation is considered. This is an important problem for the control of very high order structures where a full order controller is inappropriate.

To address this problem, an approach using the Frobenius-Hankel (FH) norm is developed. The FH norm is shown to lead to attractive robustness and performance properties. A parameter optimization method is developed which provides an iterative method for determining the FH optimal parameters of a dynamic system using a gradient approach.

Two distinct controller design methods are presented which make use of the FH optimization procedure. The first approach determines a controller which minimizes the FH norm of the closed loop system. While the solution method is iterative, the procedure proves to be straightforward to apply.

The second approach uses projective controls as a first step in the design. This allows the designer to take advantage of the attractive features of projective controls. Design


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USING THE FROBENIUS-HANKEL NORM

BY

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B. S., Hope College, 1985
M. S., University of Illinois, 1987

THESIS

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# THE DESIGN OF LOW ORDER CONTROLLERS USING THE FROBENIUS-HANKEL NORM

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University of Illinois at Urbana-Champaign, 1990
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To address this problem, an approach using the Frobenius-Hankel (FH) norm is developed. The FH norm is shown to lead to attractive robustness and performance properties. A parameter optimization method is developed which provides an iterative method for determining the FH optimal parameters of a dynamic system using a gradient approach.

Two distinct controller design methods are presented which make use of the FH optimization procedure. The first approach determines a controller which minimizes the FH norm of the closed loop system. While the solution method is iterative, the procedure proves to be straightforward to apply.

The second approach uses projective controls as a first step in the design. This allows the designer to take advantage of the attractive features of projective controls. Design parameters in the controllers are then determined by solving the FH optimization problem called the "auxiliary minimization problem."

Examples of each approach are given, including a full design problem for the control of a flexible structure using the projective controls approach. The resulting improvement in the disturbance attenuation of the system using only second order controllers points to the effectiveness of this design procedure.

## DEDICATION

To Amy. One more hurdle completed. How many more days *now* till the wedding?

## ACKNOWLEDGEMENTS

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# CHAPTER 1

# INTRODUCTION

The design of low order controllers to achieve stability, attenuation of disturbances and robustness to both modeling errors and parameter variation is an important design problem in the control of many complex systems. This is particularly true in the control of large flexible structures where factors related to cost, implementation, maintenance and reliability may make a collection of low order, decentralized controllers the preferred configuration.

The goal is thus to determine a robust, low order controller such that the closed loop system is stable and has good performance. The crucial constraint dealt with in this work is on the order of the controller which, while not necessarily predetermined, is to be of order significantly lower than the order of the system. However, the problem of determining the best such controller under these restrictions is an open problem at present.

The general control problem considered in this work is illustrated in Figure 1.1: Given the system $P(s)$, determine a low order, possibly decentralized controller $C(s)$ so that the closed loop system has good performance and robustness properties. The constraints on the order of the controller are usually dictated by limitations on the computational power of the processor which implements the controller. Decentralized controllers often are dictated when physical obstacles prohibit the availability of all the measurements at one central location.

Classical control, which relies on root locus and Bode plots, uses a one-loop-at-a-time method to design controllers. This is quite inadequate for MIMO systems of high order. Thus, for complex systems, it is common to design controllers based on optimization

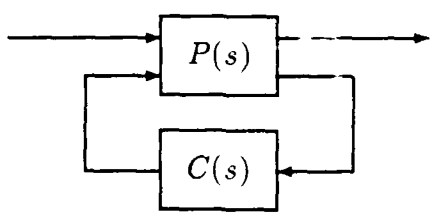


Figure 1.1: General control problem.

techniques. This approach allows complex controllers to be designed using a single criterion of optimality. The difficulty, however, lies in choosing the correct criterion and computing the resulting optimal control.

Considerable success has been realized in computing optimal controllers for norms such as $\mathcal{H}_2$ (LQG) [12] and $\mathcal{H}_\infty$ [5]. In general, closed form solutions for the optimal controller can be found in terms of the solution of Riccati equations. However, the optimal controller in general is a centralized controller of order at least as high as the plant. In many cases this is not practical. This is particularly true in the control of flexible systems where the plant's inherent high order makes the use of such controllers unacceptable.

Unfortunately, the computation of low order and decentralized optimal controllers has not been as successful as the full order centralized case. The computations of such controllers are much harder to solve since at present, there are no direct methods of computing the optimal controller as in the full order case and most methods of computing optimal controllers are either iterative or yield suboptimal controllers.

A parametric approach to the problem is to parameterize the controller and then use standard minimization techniques to optimize the parameters of the controller. Such methods have been extensively used in finding low order $\mathcal{H}_2$ optimal controllers [13,31,11,14]. Difficulties with this approach include local minima and slow convergence. Also, this method is difficult to apply in the $\mathcal{H}_\infty$ case due to the difficulty in directly computing the $\mathcal{H}_\infty$ norm.

A method for computing an optimal controller of fixed order which does not involve parameterizing the controller has been shown for the $\mathcal{H}_2$ optimal case [9] and for the bounded $\mathcal{H}_\infty$ norm case [3]. This approach has the advantage that the solution method is independent of the controller realization. However, the computation of the optimal controller involves the solution of two coupled Riccati equations and two Lyapunov equations. At present, an efficient method of solving these matrix equations is unavailable which limits its application for truly high order systems.

Because of the difficulty in computing optimal controllers, suboptimal methods are quite common due to their relative ease of computation. However, in general, these methods will not always produce near optimal solutions and may in fact produce controllers which destabilize the closed loop system.

One common suboptimal method involves the use of model reduction. Model reduction can be used to compute low order controllers in two distinct ways. One way is to approximate the plant model with a reduced order plant model and then to use a full order design method to compute the resulting lower order controller. The other way is to first compute the full order controller and then approximate the full order controller with a lower order controller. These approaches are neither optimal nor guaranteed to be stable. However, they can be significantly simpler to compute than the optimal solution. For an overview of these methods, see [1].

As can be seen, no one method exists which produces optimal or near optimal low order controllers in an efficient computational manner. In particular, no method exists to compute low order $\mathcal{H}_\infty$ optimal controllers which are attractive due to the robustness properties of the norm.

This work develops the use of the Frobenius-Hankel norm as an optimization criterion for the design of low order and decentralized controllers. This norm is established as a useful norm for this task through the examination of its analytical and numerical properties. The FH optimization approach is used in two different ways: First, to compute FH optimal controllers directly, and second, by using it to solve the subproblem of determining the free parameters of projective controllers. In both cases, the centralized and decentralized controller cases are each examined.

The organization of this work is as follows: First, a review of system theory and norms is given. Then the Frobenius-Hankel norm is introduced and its properties are explored. In particular, its relationship to the $\mathcal{H}_\infty$ and the $\mathcal{H}_2$ norm establish its robustness properties. The following chapter develops necessary conditions for an optimal FH norm solution and develops numerical methods to solve the optimization problem.

The next chapter investigates the computation of low order optimal controllers. Previously, no techniques existed for the computation of low order, $\mathcal{H}_\infty$ optimal controllers. In this chapter, necessary conditions for the solution of the $\mathcal{H}_\infty$ optimal controller are derived. However, computational problems exist in solving these conditions. This leads to the consideration of the use of FH optimization as a means of determining low order controllers. The results of the previous chapter are applied to the optimal control problem to determine necessary conditions for an optimal solution. The optimization procedure of the previous chapter is demonstrated by calculating an FH optimal controller for an example problem.

The use of projective controls is then investigated as a method of computing low order controllers. Although this is not a new technique, there are a number of differences between the published work and the presentation in this dissertation. The perspective here, is an input/output approach rather than the traditional state space approach. In particular, a number of input/output relationships are developed which give insight into the problem. A simpler formulation of the proper controller case is given which eliminates the need to transform the plant into a specific form. The FH optimization method is then applied to the problem of determining the free parameters of the projective controllers. The following chapter examines the case of decentralized projective controllers.

In order to demonstrate this approach, a design example using projective controls and FH *minimization* to determine the free parameters is presented. The control problem consists of controlling the flexible modes of a cruciform satellite structure. Using a 40-state analysis model, two decentralized, second order controllers are determined which attenuate the effect of the disturbance.

## 1.1 Notation

| | |
|---|---|
| $\mathcal{R}^n$, $\mathcal{C}^n$ | n-dimensional real and complex Euclidean spaces |
| $A^T$, $A^*$ | transpose and complex conjugate transpose of $A \in \mathcal{R}^{n\times m}$ |
| $\mathrm{Tr}\,(A)$ | trace of $A \in \mathcal{R}^{n\times n}$ , Def. 7 |
| $\lambda_i(A)$ | $i^{th}$, largest and smallest eigenvalue of $A \in \mathcal{R}^{n\times n}$ resp. |
| $\sigma_i(A), \bar{\sigma}(A), \underline{\sigma}(A)$ | $i^{th}$, largest and smallest singular value of $A \in \mathcal{R}^{n\times m}$ |
| $\mathcal{H}_2$ | space of all stable, strictly proper transfer functions |
| $\mathcal{H}_\infty$ | space of all stable, proper transfer functions |
| $\lVert G(s)\rVert_2$ | $\mathcal{H}_2$ norm of $G(s) \in \mathcal{H}_2$, Def. 4 |
| $\lVert G(s)\rVert_\infty$ | $\mathcal{H}_\infty$ norm of $G(s) \in \mathcal{H}_\infty$, Def. 5 |
| $\lVert G(s)\rVert_F$ | Frobenius-Hankel norm of $G(s) \in \mathcal{H}_2$, Def. 6 |

| | |
|---|---|
| $\sigma_i(G(s))$ | $i^{th}$ Hankel singular value of $G(s)$, Def. 3 |
| $\delta(t)$ | Dirac delta function (impulse) |
| $\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$ | transfer function, Def. 1 |
| $\mathcal{L}[\cdot]$, $\mathcal{F}[\cdot]$ | Laplace and Fourier transform operators, resp. |

# CHAPTER 2

# BACKGROUND

## 2.1 Representation of Dynamic Systems

A linear, time-invariant (LTI) system is usually represented in one of two equivalent ways. The first involves expressing the input-output relationship of the system in terms of first order differential equations. This is often called the **state space** representation of the system because of the use of the state variable. The other way to represent the system is through the Laplace transform of the impulse response of the system. This is called the **transfer function** of the system.

While both are equivalent representations of an LTI system, each has its own strengths and weaknesses as a system representation. In particular, the transfer function representation works well to represent the output of the system as a function of the system inputs. Thus, it emphasizes the input-output nature of the system. However, the transfer function of the system is defined in terms of polynomials of a complex variable. Thus, it poses computational problems for use in design and analysis. The state space approach is useful because it can easily be defined in terms of real matrices. This makes the state space system useful for computational purposes. Thus, it follows that it would be useful to be able to use the compact notation of the transfer function while retaining the state space representation for computational uses. The following notation is introduced for this purpose.

**Definition 1** *Given the state space representation of the LTI system*

$$\dot{x}(t) = Ax(t) + Bw(t) \tag{2.1}$$

$$z(t) = Cx(t) + Dw(t) \tag{2.2}$$

*where $x \in \mathcal{R}^n$ is the state, $w \in \mathcal{R}^m$ is the input, $z \in \mathcal{R}^r$ is the output, the transfer function of the system is denoted*

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \triangleq C(sI - A)^{-1}B + D. \tag{2.3}$$

Thus, through the use of the expression (2.3), the transfer function of the system is represented in terms of a state space representation using a compact notation.

### 2.1.1 Series connection

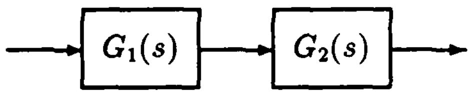


Figure 2.1: Series connection.

Given two systems $G_1(s)$ and $G_2(s)$

$$G_1(s) \triangleq \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right], \quad G_2(s) \triangleq \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \tag{2.4}$$

connected together in series as shown in Figure 2.1, the resulting system can be represented as

$$G_2(s)G_1(s) = \left[\begin{array}{cc|c} A_2 & B_2C_1 & B_2D_1 \\ 0 & A_1 & B_1 \\ \hline C_2 & D_2C_1 & D_2D_1 \end{array}\right]. \tag{2.5}$$

Note that this representation is not necessarily minimal.

### 2.1.2 Parallel connection

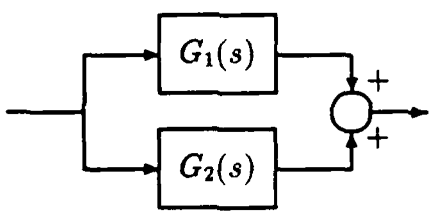


Figure 2.2: Parallel connection.

Given two systems $G_1(s)$ and $G_2(s)$ connected in parallel as in Figure 2.2, the resulting system can be represented as

$$G_1(s) + G_2(s) = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]. \tag{2.6}$$

Note that this representation is not necessarily minimal.

### 2.1.3 State space transformation

A state space transformation has no effect on the transfer function of a system. Thus, given the state space transformation $\tilde{x} = Tx$ where $T$ is a nonsingular matrix,

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right]. \tag{2.7}$$

This points out an important fact about state space representations of a system. That is, they are not unique representations of a transfer function.

### 2.1.4 Minimal representation

The minimal representation of a system is one such that it has no uncontrollable or unobservable states. If a system contains uncontrollable and unobservable states, it can be reduced to its minimal representation by removing the uncontrollable and unobservable states.

This can be easily done if the system has the following canonical form:

$$G(s) = \left[\begin{array}{cccc|c} A_{c\bar{o}} & A_{12} & A_{13} & A_{14} & B_{c\bar{o}} \\ 0 & A_{co} & A_{23} & A_{24} & B_{co} \\ 0 & 0 & A_{\bar{c}o} & A_{34} & 0 \\ 0 & 0 & 0 & A_{\bar{c}\bar{o}} & 0 \\ \hline 0 & C_{co} & C_{\bar{c}o} & 0 & D \end{array}\right]. \tag{2.8}$$

This system is equivalent to the following system

$$G(s) = \left[\begin{array}{c|c} A_{co} & B_{co} \\ \hline C_{co} & D \end{array}\right]. \tag{2.9}$$

Note that $A_{c\bar{o}}$ represents dynamics which are controllable but not observable, $A_{co}$ represents dynamics which are both controllable and observable, $A_{\bar{c}o}$ represents dynamics which are observable but not controllable, and $A_{\bar{c}\bar{o}}$ represents dynamics which are neither controllable nor observable.

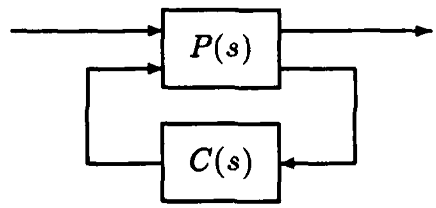


Figure 2.3: Feedback connection.

## 2.2 Feedback Systems

Given the feedback system system shown in Figure 2.3 where the plant $P(s)$ has the state space representation

$$\dot{x} = Ax + Bu + Gw \tag{2.10}$$

$$z = Hx + Eu \tag{2.11}$$

$$y = Cx + Dw \tag{2.12}$$

and $x \in \mathcal{R}^n$ is the state, $w \in \mathcal{R}^q$ is the disturbance input, $u \in \mathcal{R}^m$ is the controlled input, $z \in \mathcal{R}^s$ is the controlled output, and $y \in \mathcal{R}^r$ is the measured output, the transfer function $P(s)$ such that

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = P(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} \tag{2.13}$$

is given by

$$P(s) \triangleq \left[\begin{array}{c|cc} A & G & B \\ \hline H & 0 & E \\ C & D & 0 \end{array}\right]. \tag{2.14}$$

Note that the control system described here assumes that the controlled outputs do not contain a direct disturbance term and that the measured outputs do not contain a direct control term. These assumptions, however, are not overly restrictive since they represent common assumptions for control problems. Also, the omission of these terms does not form a fundamental restriction, but rather one made for notational simplification.

If the system $P(s)$ is controlled by

$$C(s) \triangleq \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \tag{2.15}$$

so that

$$u(s) = C(s)y(s) \tag{2.16}$$

then the closed loop system is given by

$$G(s) = \left[\begin{array}{cc|c} A + BD_cC & BC_c & G + BD_cD \\ B_cC & A_c & B_cD \\ \hline H + ED_cC & EC_c & ED_cD \end{array}\right]. \tag{2.17}$$

This expression will be used repeatedly in this work since it represents the closed loop dynamics of a general plant to a general controller.

## 2.3 Hankel Singular Values

The Hankel singular values and norms defined on Hankel singular values of a linear system have been shown to have useful applications in model reduction [23,7], in providing bounds for the $\mathcal{H}_\infty$ norm [7], and in the design of low order controllers [24,19,27].

**Definition 2** *The* **controllability** *and* **observability grammians** *of a system $G(s) \in \mathcal{H}_2$ where*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \tag{2.18}$$

*are defined as, resp.*

$$P \triangleq \int_0^\infty e^{At}BB^Te^{A^Tt}dt \tag{2.19}$$

$$Q \triangleq \int_0^\infty e^{A^Tt}C^TCe^{At}dt. \tag{2.20}$$

The grammians of the system are the unique positive definite solutions to the Lyapunov equations

$$AP + PA^T + BB^T = 0 \tag{2.21}$$

$$A^TQ + QA + C^TC = 0. \tag{2.22}$$

These equations form an efficient approach to solving for the grammians.

**Definition 3** *The* **Hankel singular values** *of $G(s) \in \mathcal{H}_2$ are given by*

$$\sigma_i(G(s)) \triangleq [\lambda_i(PQ)]^{1/2} \tag{2.23}$$

*where $P$ and $Q$ are the controllability and observability grammians of $G(s)$.*

## 2.4 Norms of Dynamic Systems

The two most common norms used for dynamic systems are the $\mathcal{H}_2$ norm and the $\mathcal{H}_\infty$ norm. These norms are reviewed here for reference in later chapters.

**Definition 4** *The $\mathcal{H}_2$ norm of $G(s) \in \mathcal{H}_2$ is defined as*

$$\|G(s)\|_2 \triangleq \left[\mathrm{Tr}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)^* G(j\omega)\, d\omega\right\}\right]^{1/2}. \tag{2.24}$$

The $\mathcal{H}_2$ norm is often called the "quadratic norm" due to its interpretation as the integral of the square of the impulse response.

**Theorem 1** *Given $G(s) \in \mathcal{H}_2$ and the* **impulse response** $g(t) = \mathcal{L}^{-1}[G(s)]$

$$\|G(s)\|_2^2 = \mathrm{Tr}\int_0^{\infty} g(t)^T g(t)\, dt. \tag{2.25}$$

The $\mathcal{H}_2$ norm can be computed from a state space representation of the system.

**Theorem 2** *Given $G(s) \in \mathcal{H}_2$ and the controllability and observability grammians $P$ and $Q$,*

$$\|G(s)\|_2^2 = \mathrm{Tr}\,\{PC^TC\} = \mathrm{Tr}\,\{QBB^T\}. \tag{2.26}$$

The $\mathcal{H}_\infty$ norm is often considered a "worst-case" norm and as such has applications to robustness issues since it can be used to bound the performance of a system.

**Definition 5** *The $\mathcal{H}_\infty$ norm of $G(s) \in \mathcal{H}_\infty$ is defined as*

$$\|G(s)\|_\infty \triangleq \sup_{\omega} \bar{\sigma}[G(j\omega)]. \tag{2.27}$$

The $\mathcal{H}_\infty$ norm is the equivalent to the maximum gain of the system as shown in the following theorem.

**Theorem 3** *Given $G(s) \in \mathcal{H}_\infty$*

$$\|G(s)\|_\infty = \sup_{w \neq 0} \frac{\|z\|_2}{\|w\|_2} \tag{2.28}$$

*where $z(t)$ is the response of $G(s)$ to the input $w(t)$.*

This theorem is an important robustness result since it states that if a bound is known on the norm of the input, a bound can be calculated for the norm of the output.

The computation of the $\mathcal{H}_\infty$ norm is a difficult problem since it cannot be computed directly. However, an upper bound for the $\mathcal{H}_\infty$ norm can be computed as follows [4,5]:

**Theorem 4** *Given $G(s) \in \mathcal{H}_\infty$ such that*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.29}$$

*and $\gamma > \bar{\sigma}(D)$. Then $\|G(s)\|_\infty < \gamma$ iff $H$ has no imaginary eigenvalues where*

$$H \triangleq \begin{bmatrix} A + BR^{-1}D^TC & \gamma^{-2}BR^{-1}B^T \\ -C^TS^{-1}C & -A^T - C^TDR^{-1}B^T \end{bmatrix} \tag{2.30}$$

*and $R \triangleq I - \gamma^{-2}D^TD$ and $S \triangleq I - \gamma^{-2}DD^T$.*

For the case where the system is strictly proper, the theorem can be simplified as follows:

**Theorem 5** *Given $G(s) \in \mathcal{H}_2$ such that*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]. \tag{2.31}$$

*Then $\|G(s)\|_\infty < \gamma$ iff $H$ has no imaginary eigenvalues where*

$$H \triangleq \begin{bmatrix} A & \gamma^{-2}BB^T \\ -C^TC & -A^T \end{bmatrix}. \tag{2.32}$$

*Equivalently, $\|G(s)\|_\infty < \gamma$ if and only if there exists an $X \geq 0$ which satisfies the Riccati equation*

$$A^TX + XA + \gamma^{-2}XBB^TX + C^TC = 0. \tag{2.33}$$

Thus, the $\mathcal{H}_\infty$ norm of a system can be computed in an iterative way by applying a search algorithm to the problem of finding the smallest $\gamma$ which satisfies the bound $\neg\|G(s)\|_\infty < \gamma$. This $\gamma$ is the $\mathcal{H}_\infty$ norm of the system.

# CHAPTER 3

# THE FROBENIUS-HANKEL NORM

Recently, Medanic and Perkins introduced the Frobenius-Hankel (FH) norm for the design of control systems [17]. The FH norm is defined as the Frobenius norm on the Hankel singular values. The motivation for its use stems from its relationship to more widely known norms such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ and its good computational properties which make it suitable for use in optimization procedures.

In this chapter, the Frobenius-Hankel norm is defined and its properties explored. In particular, both a time-domain and frequency-domain physical interpretation will made of the meaning of the FH norm and a simple computational method will be shown for calculating the FH norm. The FH norm will also be directly related to both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms.

In the following chapters the FH norm will be used as the basis for a parameter optimization problem and applied to a model reduction problem and the low order optimal controller problem.

**Definition 6** *The* **Frobenius-Hankel norm** *of* $G(s) \in \mathcal{H}_2$ *is*

$$\|G(s)\|_F \triangleq \left[\sum_{i=1}^{n} \sigma_i^2(G(s))\right]^{1/2}. \tag{3.1}$$

## 3.1 Properties of the FH Norm

The FH norm can be easily computed directly from the grammians $P$ and $Q$.

**Theorem 6** *Given the system $G(s) \in \mathcal{H}_2$ and the controllability and observability grammians $P$ and $Q$ resp., then*

$$\|G(s)\|_F^2 = \mathrm{Tr}\,\{PQ\}. \tag{3.2}$$

**Proof:** From Definition 6,

$$\|G(s)\|_F^2 = \mathrm{Tr}\,\Sigma^2 \tag{3.3}$$

where $\Sigma = diag(\sigma_1 ... \sigma_n)$. Since there exists $T$ nonsingular such that $T^{-T}PT^{-1} = \Sigma$ and $TQT^T = \Sigma$ [7]

$$\|G(s)\|_F^2 = \text{Tr}\,(T^{-T}T^T)\Sigma(TT^{-1})\Sigma \tag{3.4}$$
$$= \text{Tr}\,(T^T\Sigma T)(T^{-1}\Sigma T^{-T}) \tag{3.5}$$
$$= \text{Tr}\,PQ. \tag{3.6}$$

■

Note that this expression involves the solution of two Lyapunov equations but avoids computation of the Hankel singular values. This expression for the FH norm is important because the trace function is more well-behaved than the singular value function.

### 3.1.1 Time-domain properties

A time-domain interpretation of the Frobenius-Hankel norm is as follows:

**Theorem 7** *Given the system $G(s) \in \mathcal{H}_2$ and the impulse response of the system $g(t)$, then*

$$\|G(s)\|_F^2 = \text{Tr} \int_0^\infty t\, g(t)^T g(t)\, dt. \tag{3.7}$$

**Proof:** From Theorem 6,

$$\|G(s)\|_F^2 = \text{Tr}\,PQ. \tag{3.8}$$

From Definition 3,

$$\text{Tr}\,PQ = \lim_{T\to\infty} \text{Tr} \left[\int_0^T e^{At}BB^Te^{A^Tt}\,dt\right]\left[\int_0^T e^{A^T\tau}C^TCe^{A\tau}\,d\tau\right] \tag{3.9}$$

which is equivalent to

$$\text{Tr}\,PQ = \lim_{T\to\infty} \text{Tr} \int_0^T \int_0^T \left[Ce^{A(t+\tau)}B\right]\left[Ce^{A(t+\tau)}B\right]^T dt\,d\tau. \tag{3.10}$$

Let $g(\tau) \triangleq Ce^{A\tau}B$,

$$\text{Tr}\,PQ = \lim_{T\to\infty} \text{Tr} \int_0^T \int_0^T g(t+\tau)^T g(t+\tau)\,dt\,d\tau \tag{3.11}$$

$$\text{Tr}\,PQ = \lim_{T\to\infty} \text{Tr} \int_0^T \int_\tau^{T+\tau} g(t)^T g(t)\,dt\,d\tau \tag{3.12}$$

Let $H(\tau) \triangleq \int_\tau^{T+\tau} g(t)^T g(t)\, dt$.

$$\mathrm{Tr}\, PQ = \lim_{T\to\infty} \mathrm{Tr} \int_0^T H(\tau)\, d\tau. \tag{3.13}$$

Integrating by parts,

$$\mathrm{Tr}\, PQ = \lim_{T\to\infty} \mathrm{Tr} \left[ H(\tau)\tau|_0^T - \int_0^T \tau\, dH(\tau) \right] \tag{3.14}$$

$$\mathrm{Tr}\, I\, Q = \lim_{T\to\infty} \mathrm{Tr} \left[ \int_0^T (T-t) g(t+T)^T g(t+T) + t g(t)^T g(t)\, dt \right]. \tag{3.15}$$

In the limit as $T \to \infty$, $g(t+T) \to 0$, thus

$$\mathrm{Tr}\, PQ = \mathrm{Tr} \int_0^\infty t g(t)^T g(t)\, dt. \tag{3.16}$$

■

Note the similarity of this expression to the one given in Theorem 1. In fact, a comparison of the two expressions shows that the FH norm is in fact equivalent to a time-weighted $\mathcal{H}_2$ norm. The weighting factor $t$ acts to weight relatively more heavily the impulse response at later times than at earlier times. Thus, large initial responses are weighted less, but lightly damped dynamics are weighted more.

### 3.1.2 Frequency-domain properties

**Theorem 8** *Given the system $G(s) \in \mathcal{H}_2$ and the frequency response of the system $G(j\omega) = G(s)|_{s=j\omega}$, then*

$$\|G(s)\|_F^2 = \frac{j}{2\pi} \mathrm{Tr} \int_{-\infty}^{\infty} \frac{dG(j\omega)}{d\omega} G(j\omega)^*\, d\omega. \tag{3.17}$$

**Proof:** Applying Parseval's Theorem to (3.7) yields

$$\begin{aligned} \|G(s)\|_F^2 &= \frac{1}{2\pi} \mathrm{Tr} \int_{-\infty}^{\infty} \mathcal{F}[tg(t)] \mathcal{F}[g(t)]^*\, d\omega && (3.18) \\ &= \frac{1}{2\pi} \mathrm{Tr} \int_{-\infty}^{\infty} j \left( \frac{dG(j\omega)}{d\omega} \right) G(j\omega)^*\, d\omega && (3.19) \end{aligned}$$

■

Note that the this expression shows that there are two components to the FH norm. The first component comes from the $\frac{dG(j\omega)}{d\omega}$ term in the above expression which implies that the FH norm weights the "flatness" of the transfer functions. The second component is due to the $G(j\omega)$ term in the above expression which implies that the FH norm also weights the magnitude of the transfer function.

## 3.2 Relationships with Other Norms

Although a physical interpretation has been given for the Frobenius-Hankel norm, relating it to familiar norms is useful in providing further motivation.

The Frobenius-Hankel norm can be related to the $\mathcal{H}_\infty$ norm through the Hankel singular values of the system. In the following two theorems, expressions are shown which relate the two norms using the Hankel singular values.

This first result defines an interval in which both the $\mathcal{H}_\infty$ and FH norm must lie. Thus, results which apply to the $\mathcal{H}_\infty$ norm can be applied to the FH norm through this result.

**Theorem 9** *Given* $G(s) \in \mathcal{H}_2$,

$$\bar{\sigma}(G(s)) \leq \|G(s)\|_\infty \leq 2\sum_{i=1}^{n} \sigma_i(G(s)) \tag{3.20}$$

$$\bar{\sigma}(G(s)) \leq \|G(s)\|_F \leq \sum_{i=1}^{n} \sigma_i(G(s)). \tag{3.21}$$

**Proof:** The first expression has been shown in [6] and [7]. To show the second expression, begin first with the upper bound.

$$\left(\sum_{i=1}^{n} \sigma_i\right)^2 = \left(\sum_{i=1}^{n} \sigma_i\right)\left(\sum_{j=1}^{n} \sigma_j\right) \tag{3.22}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i\sigma_j \tag{3.23}$$

$$= \sum_{k=1}^{n} \sigma_k^2 + \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} \sigma_i\sigma_j \tag{3.24}$$

$$\geq \sum_{k=1}^{n} \sigma_k^2 = \|G(s)\|_F^2. \tag{3.25}$$

The lower bound follows directly from

$$\|G(s)\|_F^2 = \sum_{i=1}^{n} \sigma_i^2 \geq \bar{\sigma}. \tag{3.26}$$

■

Using the results of the previous theorem, the $\mathcal{H}_\infty$ norm can be bounded in terms of the FH norm. This gives a measure of the "closeness" of the $\mathcal{H}_\infty$ norm to the FH norm.

**Theorem 10** *Given* $G(s) \in \mathcal{H}_2$,

$$\frac{1}{\sqrt{n}}\|G(s)\|_F \leq \|G(s)\|_\infty \leq 2\sqrt{n}\|G(s)\|_F. \tag{3.27}$$

**Proof:** To show the lower bound, first note that

$$\sum_{i=1}^{n} \sigma_i^2 \leq n\bar{\sigma}^2. \tag{3.28}$$

Then it follows from (3.20) that

$$\|G(s)\|_\infty \geq \bar{\sigma} \geq \frac{1}{\sqrt{n}}\|G(s)\|_F \tag{3.29}$$

which proves the lower bound. To show the upper bound, note that

$$\sum_{i=1}^{n} \sigma_i = E^T\Sigma \tag{3.30}$$

where $E^T \triangleq \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$ and $\Sigma^T \triangleq \begin{bmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_n \end{bmatrix}$. The Frobenius norm of a matrix is defined as

$$\|X\|_{Fr} \triangleq [\mathrm{Tr}\, A^T A]^{1/2}. \tag{3.31}$$

Since the Frobenius norm is a valid matrix norm

$$\begin{aligned} \|E^T\Sigma\|_{Fr} &\leq \|E^T\|_{Fr}\|\Sigma\|_{Fr} && (3.32)\\ &= [\mathrm{Tr}\, E^T E]^{1/2}[\mathrm{Tr}\, \Sigma^T\Sigma]^{1/2} && (3.33)\\ &= \sqrt{n}\|G(s)\|_F. && (3.34) \end{aligned}$$

However, since $E^T\Sigma$ is a positive valued scalar,

$$\|E^T\Sigma\|_{Fr} = E^T\Sigma. \tag{3.35}$$

Thus,

$$\sum_{i=1}^{n} \sigma_i \leq \sqrt{n}\|G(s)\|_F. \tag{3.36}$$

Substituting into (3.20) yields the upper bound. ■

These expressions establish the robustness properties of the FH norm through the $\mathcal{H}_\infty$ norm. Thus, since the computation of the FH norm is much simpler, one rationale for the use of the FH norm is established.

The Frobenius-Hankel norm can also be related to the sensitivity of the $\mathcal{H}_2$ norm to a shift of the eigenvalues of the system along the real axis. Such a shift in the eigenvalues can be represented by letting $A$ be given by

$$A(\alpha) = A_o + \alpha I. \tag{3.37}$$

Thus, $\lambda(A) = \lambda(A_o) + \alpha$ and the sensitivity of the $\mathcal{H}_2$ norm to a shift in the eigenvalues along the real axis is given by $\frac{d}{d\alpha}\|G(s)\|_2^2$. The following theorem relates this sensitivity to the FH norm.

**Theorem 11** *Let the eigenvalues of the system $G(s) \in \mathcal{H}_2$ be given by $\lambda_i = \bar{\lambda}_i + \alpha$, then*

$$\left.\frac{d}{d\alpha}\|G(s)\|_2^2\right|_{\alpha=0} = 2\|G(s)\|_F^2. \tag{3.38}$$

**Proof:** Let

$$J \triangleq \|G(s)\|_2^2 = \mathrm{Tr}\, PC^TC \tag{3.39}$$

where

$$AP + PA^T + BB^T = 0. \tag{3.40}$$

Then

$$\frac{dJ}{d\alpha} = \mathrm{Tr}\, P_\alpha C^TC \tag{3.41}$$

where $P_\alpha \triangleq \frac{dP}{d\alpha}$ satisfies

$$AP_\alpha + P_\alpha A^T + 2P = 0. \tag{3.42}$$

Let $Q$ satisfy

$$QA + A^TQ + C^TC = 0. \tag{3.43}$$

Then, using the properties of the trace, it can be shown that

$$\mathrm{Tr}\, P_\alpha C^TC = 2\mathrm{Tr}\, PQ. \tag{3.44}$$

Thus

$$\frac{dJ}{d\alpha} = 2\mathrm{Tr}\, PQ. \tag{3.45}$$

■

This expression establishes the FH norm as a sensitivity measure for the $\mathcal{H}_2$ norm with respect to a change in the relative stability of the system. This is an important relationship since in flexible structures problems, often the damping of the system is not known well. Also, a change in the relative stability of a lightly damped structure is the variation most likely to cause instability.

# CHAPTER 4

# FROBENIUS-HANKEL NORM OPTIMIZATION

In this chapter a common framework for solving optimal FH norm problems will be presented. In later chapters, specific problems will be solved under this framework.

The general problem to be solved is: given a parameterized system, find the set of parameters which minimize the FH norm of the system. The first section involves finding necessary conditions for such a set of optimal parameters. Solutions to these necessary conditions are explored in the following sections. The final section applies the general FH optimization problem to the problem of FH optimal model reduction.

## 4.1 The FH Optimization Problem

The FH optimization problem is: Let the parameterized system $G(s) \in \mathcal{H}_2$ be given by

$$G(s) = \left[\begin{array}{c|c} A(\xi) & B(\xi) \\ \hline C(\xi) & 0 \end{array}\right] \tag{4.1}$$

where $\xi$ is the parameter matrix. The FH norm of $G(s)$ can be computed as

$$J = \|G(s)\|_F^2 = \mathrm{Tr}\,\{PQ\} \tag{4.2}$$

where $P$ and $Q$ satisfy

$$0 = AP + PA^T + BB^T \tag{4.3}$$

$$0 = A^TQ + QA + C^TC. \tag{4.4}$$

The optimization problem is thus, find $\xi$ such that the criterion (4.2) is minimized subject to the constraints (4.3)-(4.4).

## 4.2 General Solution Methods

This constrained optimization problem can be converted to an unconstrained optimization problem using Lagrange multipliers. The augmented criterion is given by

$$\hat{J} = \mathrm{Tr}\,\{PQ + L(AP + PA^T + BB^T) + M(A^TQ + QA + C^TC)\}. \tag{4.5}$$

Using this approach, necessary conditions for an optimal solution are

$$\frac{\partial \hat{J}}{\partial P} = A^T L + LA + Q = 0 \tag{4.6}$$

$$\frac{\partial \hat{J}}{\partial Q} = AM + MA^T + P = 0 \tag{4.7}$$

$$\frac{\partial \hat{J}}{\partial L} = AP + PA^T + BB^T = 0 \tag{4.8}$$

$$\frac{\partial \hat{J}}{\partial M} = A^T Q + QA + C^T C = 0 \tag{4.9}$$

$$\frac{\partial \hat{J}}{\partial \xi} = \frac{\partial}{\partial \xi} \mathrm{Tr}\,\{2A^T(LP + QM) + B^T LB + C^T CM\} = 0. \tag{4.10}$$

In general these equations can not be solved for the optimal $\xi$ directly. However, iterative methods may be applied to this problem.

### 4.2.1 A steepest descent approach

A steepest descent approach to this problem is to find the direction of steepest descent and to take a step in that direction. The direction of steepest descent is in the direction of the gradient wrt $\xi$. The gradient of $J$ wrt to $\xi$ is given by

$$\frac{dJ}{d\xi} = \frac{\partial \hat{J}}{\partial \xi} + \frac{\partial \hat{J}}{\partial P}\frac{dP}{d\xi} + \frac{\partial \hat{J}}{\partial Q}\frac{dQ}{d\xi} + \frac{\partial \hat{J}}{\partial L}\frac{dL}{d\xi} + \frac{\partial \hat{J}}{\partial M}\frac{dM}{d\xi}. \tag{4.11}$$

If $P, Q, L, M$ satisfy (4.6)-(4.9), then

$$\frac{dJ}{d\xi} = \frac{\partial \hat{J}}{\partial \xi}. \tag{4.12}$$

The parameter update is given by

$$\xi_{i+1} = \xi_i - \epsilon \frac{dJ}{d\xi}. \tag{4.13}$$

This method was is illustrated in Figure 4.1 and used in [27] to solve the FH optimization subproblem.

### 4.2.2 Riccati equation approach

The Riccati approach [24] uses Riccati equations instead of Lyapunov equations. The Riccati equations are constructed so that the iterative solution converges to the solution

1. Select $\xi_o$ so that $A(\xi_o)$ is stable.

2. Let $i = 1$.

3. Solve Eqs. (4.6)-(4.9) for $L$, $M$, $P$ and $Q$.

4. Calculate $\xi_{i+1}$ from Eqs. (4.12) and (4.13).

5. If the parameters have not converged, let $i = i + 1$ and go to 2.

Figure 4.1: Steepest descent algorithm.

of the Lyapunov equations. The advantage of this approach is that the equations can be solved for all systems, stable or unstable. Thus, an initial stabilizing controller is not needed. However, the drawback to this approach lies in the fast that Riccati equations are computationally much more time-consuming to solve than Lyapunov equations.

The iterative equations are

$$A^T L_{i+1} + L_{i+1} A - L_{i+1} R L_{i+1} + L_i R L_i + Q = 0 \tag{4.14}$$

$$A M_{i+1} + M_{i+1} A^T - M_{i+1} R M_{i+1} + M_i R M_i + P = 0 \tag{4.15}$$

$$A P_{i+1} + P_{i+1} A^T - P_{i+1} R P_{i+1} + P_i R P_i + B B^T = 0 \tag{4.16}$$

$$A^T Q_{i+1} + Q_{i+1} A - Q_{i+1} R Q_{i+1} + Q_i R Q_i + C^T C = 0 \tag{4.17}$$

where $\xi_i$ is the solution of

$$\frac{\partial \hat{J}}{\partial \xi} = 0. \tag{4.18}$$

Note that if this iterative algorithm converges, it converges to the solution of the corresponding Lyapunov equations. Thus, a solution of these equations satisfies the necessary conditions for an optimal solution.

## 4.3 FH Optimal Model Reduction

The model reduction problem is an important one since it is often necessary to approximate a high order model with a lower order model. The use of a lower order approximation

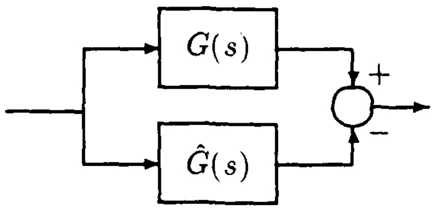


Figure 4.2: Model reduction problem.

makes design and analysis simpler although with a less accurate model.

The model reduction problem is as follows: Given an $n$-th order system

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{4.19}$$

find a $k$-th order approximation

$$\hat{G}(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \tag{4.20}$$

that minimizes the FH norm of the error $\|G(s) - \hat{G}(s)\|_F$.

The error system as illustrated in Figure 4.2 is

$$G(s) - \hat{G}(s) = \left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & D_e \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & \hat{A} & \hat{B} \\ \hline C & -\hat{C} & D - \hat{D} \end{array}\right]. \tag{4.21}$$

Note that the parameters of the error system are simply the matrices that define the reduced order model. Thus,

$$\xi = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}. \tag{4.22}$$

In order to apply the results for FH optimization, the system to be optimized must be strictly proper. Thus, for the case of model reduction, the error system must be strictly proper. This is satisfied if and only if $D_e = 0$. Thus,

$$\hat{D} = D. \tag{4.23}$$

In order to simplify the solution of this problem, note that the matrices which define the error system can be written as a linear function of the parameters $\xi$. Thus,

$$A_e = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} = A_o + T\xi T \tag{4.24}$$

$$B_e = \begin{bmatrix} 0 \\ B \end{bmatrix} + \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} 0 \\ I_m \end{bmatrix} = B_o + T\xi B_1 \tag{4.25}$$

$$C_e = \begin{bmatrix} 0 & C \end{bmatrix} + \begin{bmatrix} 0 - I_r \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} = C_o + C_1\xi T \tag{4.26}$$

$$D_e = D + \begin{bmatrix} 0 & -I_r \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} 0 \\ I_m \end{bmatrix} = D_o + C_1\xi B_1. \tag{4.27}$$

This greatly simplifies the necessary conditions since the parameters are given by a single matrix.

The necessary conditions from (4.6)-(4.9) are

$$\frac{\partial \hat{J}}{\partial L} = A_e P + P A_e^T + B_e B_e^T = 0 \tag{4.28}$$

$$\frac{\partial \hat{J}}{\partial M} = A_e^T Q + Q A_e + C_e^T C_e = 0 \tag{4.29}$$

$$\frac{\partial \hat{J}}{\partial P} = A_e^T L + L A_e + Q = 0 \tag{4.30}$$

$$\frac{\partial \hat{J}}{\partial Q} = A_e M + M A_e^T + P = 0. \tag{4.31}$$

Since the parameters $\xi$ are given by (4.22), (4.10) becomes

$$\frac{\partial \hat{J}}{\partial \xi} = 2[T(LP + QM)T + TLB_e B_1^T + C_1^T C_e M T = 0. \tag{4.32}$$

The FH optimal model reduction could thus be solved by applying one of the methods discussed in the previous sections.

# CHAPTER 5

# THE $\mathcal{H}_\infty$ AND FH OPTIMAL CONTROLLER

In this chapter, two methods of computing low order optimal controllers are developed. First, necessary conditions for an $\mathcal{H}_\infty$ optimal controller are derived. These represent new results in the determination of low order $\mathcal{H}_\infty$ optimal controllers. However, computational problems make this approach difficult.

The second method developed is FH optimal control. This method is chosen due to its robustness properties as shown in Chapter 3 and its good numerical properties.

## 5.1 The $\mathcal{H}_\infty$ Optimal Output Feedback Controller

The use of the $\mathcal{H}_\infty$ norm is motivated by robustness concerns. The $\mathcal{H}_\infty$ norm of a system describes the maximum gain from input to output of the system. Thus, it gives the "worst-case" performance of the system. This is very useful particularly when very little is know about the input signal. The $\mathcal{H}_\infty$ norm also is useful since it can be used to guarantee stability of the system despite uncertainty in the model of the system.

This problem has been solved for the state feedback and full order feedback cases [5]. Also, sufficient conditions for computing a low order controller which satisfies an upper bound on the $\mathcal{H}_\infty$ norm of the system have been shown [3,2]. The goal of this chapter is to determine necessary conditions for the solution of the $\mathcal{H}_\infty$ optimal control problem.

The $\mathcal{H}_\infty$ optimal control problem is as follows: Given the plant $P(s)$, determine the controller $C(s)$ which minimizes the $\mathcal{H}_\infty$ norm of the closed loop system $G(s)$.

Let the plant have the form

$$P(s) = \left[\begin{array}{c|cc} A & G & B \\ \hline H & 0 & E \\ C & 0 & 0 \end{array}\right] \tag{5.1}$$

where the controlled outputs have no cross terms, i.e.,

$$\begin{bmatrix} H & E \end{bmatrix}^T \begin{bmatrix} H & E \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}. \tag{5.2}$$

The controller is assumed to be a static controller. Thus,

$$C(s) = K. \tag{5.3}$$

The closed loop system then is

$$G(s) = \left[\begin{array}{c|c} A + BKC & G \\ \hline H + EKC & 0 \end{array}\right] = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]. \tag{5.4}$$

### 5.1.1 State feedback controllers

The problem of determining the $\mathcal{H}_\infty$ optimal controller in the case of state feedback ($C = I$) has been shown in [5]. The results are repeated here for comparison with the following section.

**Theorem 12** *The $\mathcal{H}_\infty$ bound $\|G(s)\|_\infty < \gamma$ if and only if there exists $M > 0$ such that*

$$A^T M + MA + Q + \gamma^{-2} MGG^T M - MBR^{-1}B^T M = 0. \tag{5.5}$$

*If such an $M$ exists, then one controller which satisfies the $\mathcal{H}_\infty$ bound is*

$$C(s) = K \tag{5.6}$$

*where*

$$K = -R^{-1}B^T M. \tag{5.7}$$

Note that the conditions in the theorem are necessary and sufficient. Thus, the controller which corresponds to the smallest $\gamma$ which satisfies the conditions of the theorem is the $\mathcal{H}_\infty$ optimal controller. Thus, the state feedback $\mathcal{H}_\infty$ optimal control problem can be solved by using a search algorithm on $\gamma$ to determine the minimum.

### 5.1.2 Output feedback controllers

The case for the case of output feedback controllers is not as simple. This thesis represents the first work on $\mathcal{H}_\infty$ optimal output feedback controllers. Previous work focused only on sufficient conditions for an $\mathcal{H}_\infty$ bound to hold.

From Theorem 5, the bound $\|G(s)\|_\infty < \gamma$ holds iff there exists $M \geq 0$ such that

$$\tilde{A}^T M + M\tilde{A} + \gamma^{-2} M\tilde{B}\tilde{B}^T M + \tilde{C}^T\tilde{C} = 0. \tag{5.8}$$

Thus, the $\mathcal{H}_\infty$ optimal control problem reduces to the constrained minimization problem

$$\inf_K J = \gamma^2 \tag{5.9}$$

subject to (5.8) and $M \geq 0$.

Using Lagrange multipliers, the equivalent unconstrained minimization problem is

$$\inf_K \hat{J} = \gamma^2 + \mathrm{Tr}\, L(\tilde{A}^T M + M\tilde{A} + \gamma^{-2} M\tilde{B}\tilde{B}^T M + \tilde{C}^T\tilde{C}) + \sum_{i=1}^{n} n_i \lambda_i(M). \tag{5.10}$$

The necessary conditions for an optimal solution are the Kuhn-Tucker conditions given by

$$n_i \lambda_i(M) = 0, \quad n_i \geq 0 \tag{5.11}$$

and

$$\frac{\partial \hat{J}}{\partial M} = (\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T M)L + L(\tilde{A} + \gamma^{-2}\tilde{B}\tilde{B}^T M)^T + XNX^{-1} = 0 \tag{5.12}$$

$$\frac{\partial \hat{J}}{\partial L} = \tilde{A}^T M + M\tilde{A} + \gamma^{-2} M\tilde{B}\tilde{B}^T M + \tilde{C}^T\tilde{C} = 0 \tag{5.13}$$

$$\frac{\partial \hat{J}}{\partial K} = 2RB^T MLC^T + 2KCLC^T = 0 \tag{5.14}$$

$$\frac{\partial \hat{J}}{\partial \gamma^2} = 1 - \gamma^{-4}\mathrm{Tr}\, LMGG^T M = 0 \tag{5.15}$$

where $N \triangleq diag\{n_i\}$ and $X$ satisfies $MX = X\Lambda$ with $\Lambda \triangleq diag\{\lambda_i(M)\}$.

The controller parameter $K$ can be eliminated by introducing the projection matrix $\tau$. From (5.14),

$$K = -R^{-1}B^T MLC^T(CLC^T)^{-1}. \tag{5.16}$$

Then (5.13) becomes

$$A^T M + MA + \gamma^{-2}MGG^T M + Q - MBR^{-1}B^T M + \tau_\perp^T MBR^{-1}B^T M\tau_\perp = 0 \tag{5.17}$$

where

$$\tau \triangleq LC^T(CLC^T)^{-1}C \tag{5.18}$$

$$\tau_\perp \triangleq I - \tau. \tag{5.19}$$

From (5.15), an expression for $\gamma$ may be derived:

$$\gamma^2 = [\mathrm{Tr}\, LMGG^T M]^{\frac{1}{2}}. \tag{5.20}$$

However, this expression is not useful for determining the optimal $\gamma$ since $L$ cannot be determined uniquely. Note that if $L = L_o$ is a solution to (5.12), then $L = \alpha L_o$ is also a solution. Thus, the optimal controller is the controller associated with the minimum $\gamma$ such that the necessary conditions have a solution.

Note that from this expression one can conclude that if the minimum $\gamma$ is non-zero, then $L \neq 0$. But from (5.12), this is only possible if $A + \gamma^{-2} MGG^T M$ has purely imaginary eigenvalues or $M$ becomes indefinite. Thus, the minimal $\gamma$ can be found by reducing the value of $\gamma$ until one of the stopping conditions is violated. The stopping conditions are

1. $A + \gamma^{-2} MGG^T M$ has no purely imaginary eigenvalues

2. $M \geq 0$

In the state feedback case, $C$ is full rank and thus $\tau = I$. This decouples (5.13) and (5.12). Thus (5.13) becomes

$$A^T M + MA + \gamma^{-2} MGG^T M + Q - MBR^{-1} B^T M = 0. \qquad (5.21)$$

Note that this agrees with the results for the state feedback case given in [5].

One possible algorithm for solving this problem is given in Figure 5.1. This algorithm has proved to have slow convergence properties which become more acute as the minimum is approached. However, efficient algorithms for the solution of these equations will require additional investigation beyond the scope of this work.

### 5.2 The FH Optimal Controller

This section derives the necessary conditions for an FH optimal controller. Note that the assumption of the use of a time-invariant controller is in itself a restriction. Appendix B shows that the unconstrained controller is a time-varying controller. Thus, the assumption of a time-invariant controller is in itself a constraint and results in a higher FH norm than in the unconstrained case.

The FH optimal control problem is as follows: Given the plant $P(s)$ determine the controller $C(s)$ which minimizes the $\mathcal{H}_\infty$ norm of the closed loop system $G(s)$.

1. Let $\tau = I$, $\gamma = \infty$.
2. Find smallest $\gamma$ such that (5.17) has a solution.
3. Solve for $L$ from (5.12).
4. Iteratively solve (5.17) and (5.18)-(5.19) for $\tau$ and $M$.
5. Go to 2.

Figure 5.1: An $\mathcal{H}_\infty$ optimal controller algorithm

Let the plant have the form

$$P(s) = \left[\begin{array}{c|cc} A & G & B \\ \hline H & 0 & E \\ C & D & 0 \end{array}\right]. \tag{5.22}$$

The controller is assumed to be a $p$-th order dynamic controller. Thus,

$$C(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]. \tag{5.23}$$

The closed-loop system is

$$G(s) = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] = \left[\begin{array}{cc|c} A + BD_cC & BC_c & G + BD_cD \\ B_cC & A_c & B_cD \\ \hline H + ED_cC & EC_c & ED_cD \end{array}\right]. \tag{5.24}$$

The parameters of the closed loop system are the matrices that define the controller. Thus,

$$\xi = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}. \tag{5.25}$$

In order to simplify the notation, the following expressions are used:

$$\tilde{A} = \hat{A} + \hat{B}\xi\hat{C} \tag{5.26}$$

$$\tilde{B} = \hat{G} + \hat{B}\xi\hat{D} \tag{5.27}$$

$$\tilde{C} = \hat{H} + \hat{E}\xi\hat{C} \tag{5.28}$$

$$\tilde{D} = \hat{E}\xi\hat{D} \tag{5.29}$$

where

$$\begin{gathered}\hat{A} = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \quad \hat{B} = \begin{bmatrix} I_p & 0 \\ 0 & B \end{bmatrix} \quad \hat{C} = \begin{bmatrix} I_p & 0 \\ 0 & C \end{bmatrix} \\ \hat{D} = \begin{bmatrix} 0 \\ D \end{bmatrix} \quad \hat{E} = \begin{bmatrix} 0 & E \end{bmatrix} \quad \hat{G} = \begin{bmatrix} 0 \\ G \end{bmatrix} \\ \hat{H} = \begin{bmatrix} 0 & H \end{bmatrix}.\end{gathered} \tag{5.30}$$

In order for the closed-loop system to be strictly proper it is required that $\tilde{D} = 0$. Thus, one condition for the optimal solution is

$$\hat{E}\hat{D}_c\hat{D} = 0. \tag{5.31}$$

The necessary conditions for an optimal solution from (4.6)-(4.9) are

$$\frac{\partial \hat{J}}{\partial L} = \tilde{A}P + P\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 \tag{5.32}$$

$$\frac{\partial \hat{J}}{\partial M} = \tilde{A}^TQ + Q\tilde{A} + \tilde{C}^T\tilde{C} = 0 \tag{5.33}$$

$$\frac{\partial \hat{J}}{\partial P} = \tilde{A}^TL + L\tilde{A} + Q = 0 \tag{5.34}$$

$$\frac{\partial \hat{J}}{\partial Q} = \tilde{A}M + M\tilde{A}^T + P = 0 \tag{5.35}$$

Equation (4.10) becomes

$$\frac{\partial \hat{J}}{\partial \xi} = 2[\hat{B}^T(LP + QM)\hat{C}^T + \hat{B}^TL(\hat{G} + \hat{B}\xi\hat{D})\hat{D}^T + \hat{E}^T(\hat{H} + \hat{E}\xi\hat{C})M\hat{C}^T]. \tag{5.36}$$

If, in addition, the controller is non-dynamic, i.e., $C(s) = D_c$, then the closed loop system is

$$G(s) = \left[\begin{array}{c|c} A + BD_cC & G \\ \hline H + ED_cC & 0 \end{array}\right] = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]. \tag{5.37}$$

The necessary conditions for an optimal control reduce to (5.32)-(5.35) and

$$\frac{\partial \hat{J}}{\partial D_c} = 2[B^T(LP + QM)C^T + B^TL(G + BD_cD)D^T + E^T(H + ED_cC)MC^T]. \tag{5.38}$$

Satisfying the condition $ED_cD = 0$ leads to three special cases:

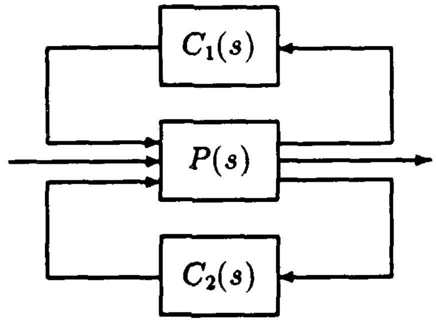


Figure 5.2: Plant with decentralized controllers

1. Noise-Free Measurements ($D = 0$)

2. Cheap Control ($E = 0$)

3. Strictly Proper Controller ($D_c = 0$)

If the measured outputs of the plant do not contain a direct feedthrough term for the disturbance, then $D = 0$. This is referred to as the "noise-free" case since the measured outputs are considered to be uncorrupted by any disturbance. If the controlled outputs of the plant do not include a control term, then $E = 0$. This is referred to as the "cheap control" case since the application of large amounts of control energy is not reflected in the controlled outputs. If the measured outputs include a non-singular noise term ($DD^T$) and the controlled outputs include a non-singular control term ($E^T E$), then $D_c = 0$. In this case, the resulting controller is strictly proper.

Of course (5.31) can be satisfied in a combination of these three cases. This would imply that some channels of the control would be strictly proper, some noise-free and some with cheap control.

## 5.3 FH Optimal Decentralized Controllers

The decentralized control problem shown in Figure 5.2 arises when all the controls and measurements are not available at one central location. Thus, each separate channel has a separate controller which works with the available measurements and controls.

The FH optimal decentralized control problem is: Given a system $P(s)$ whose measurements and controls have been decentralized into $l$-channels controlled by $l$ decentralized, dynamic controllers, find the $l$ controllers $C_i(s)$ which minimize the Frobenius-Hankel norm of the closed loop system $G(s)$.

Assume $P(s)$ has the form

$$P(s) = \left[\begin{array}{c|cccc} A & G & B_1 & \dots & B_l \\ \hline H & 0 & E_1 & \dots & E_l \\ C_1 & D_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_l & D_l & 0 & \dots & 0 \end{array}\right] \tag{5.39}$$

and the $l$ controllers are given by

$$C_i(s) = \left[\begin{array}{c|c} A_{ci} & B_{ci} \\ \hline C_{ci} & D_{ci} \end{array}\right]. \tag{5.40}$$

The solution to this problem is to reformulate it in terms of the centralized control problem. Thus, define the centralized controller $C(s)$ in terms of the decentralized controllers $C_i(s)$.

$$C(s) = \begin{bmatrix} C_1(s) & & 0 \\ & \ddots & \\ 0 & & C_l(s) \end{bmatrix} = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \tag{5.41}$$

where

$$A_c = diag(A_{c1}, ..., A_{cl}) \tag{5.42}$$

$$B_c = diag(B_{c1}, ..., B_{cl}) \tag{5.43}$$

$$C_c = diag(C_{c1}, ..., C_{cl}) \tag{5.44}$$

$$D_c = diag(D_{c1}, ..., D_{cl}). \tag{5.45}$$

Also, let

$$C = \begin{bmatrix} C_1 \\ \vdots \\ C_l \end{bmatrix}, \quad D = \begin{bmatrix} D_1 \\ \vdots \\ D_l \end{bmatrix} \tag{5.46}$$

$$B = \begin{bmatrix} B_1 & \dots & B_l \end{bmatrix} \tag{5.47}$$

$$E = \begin{bmatrix} E_1 & \dots & E_l \end{bmatrix}. \tag{5.48}$$

Thus, the decentralized optimal control problem is identical to the centralized case except that the parameters of the controller are constrained. Necessary conditions for an optimal solution are thus (5.32)-(5.35) with the additional parameter constraints given by (5.42)-(5.45).

## 5.4 Example

In order to demonstrate the computation of an FH optimal controller, an example control problem is presented.

### 5.4.1 Problem definition

The optimization problem is: Given the plant

$$P(s) = \left[\begin{array}{cccc|cc} -0.4335 & -0.0118 & -0.9231 & -0.4643 & 0.8854 & -0.7382 \\ -0.9160 & -0.5185 & -0.4110 & -0.0779 & 0.1747 & 1.5473 \\ -0.0414 & -0.6085 & -0.7507 & -0.8901 & -1.4939 & 0.8204 \\ -0.4828 & -0.0916 & -0.2014 & -0.9215 & -1.1423 & -1.5361 \\ \hline 0.9782 & 1.9938 & -0.8140 & -0.8819 & 0 & -1.5443 \\ 0.1821 & 0.3387 & 1.6250 & 1.0326 & 0 & 0 \end{array}\right] \tag{5.49}$$

determine a second order, proper, stabilizing controller $C(s)$ which minimizes the FH norm of the closed loop system.

### 5.4.2 Controller optimization

To start the algorithm, an initial, non-zero, stabilizing controller is needed. Such a controller is given by

$$C(s) = \left[\begin{array}{cc|c} -1 & 0 & -0.0118 \\ 0 & -1 & -0.0555 \\ \hline 0.0846 & -0.1728 & 0 \end{array}\right]. \tag{5.50}$$

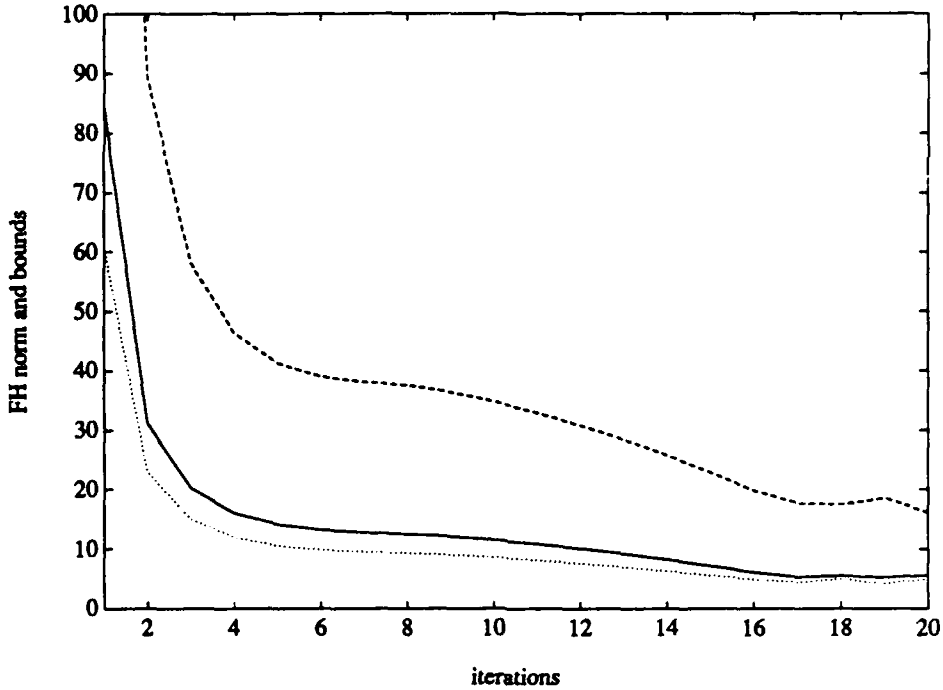


Figure 5.3: Iteration history of FH norm of the system.

Using the Matlab routine given in Appendix C, the optimal controller was computed. Figure 5.3 shows the FH norm at each iteration in the algorithm along with the $\mathcal{H}_\infty$ bounds of the system given in (3.20). In this example, the FH norm was reduced from its initial value of 84.9 down to 5.6. and the upper bound on the $\mathcal{H}_\infty$ norm reduced from 240 down to 20. The optimal controller is determined to be

$$C(s) = \left[\begin{array}{cc|c} -0.7966 & -0.2337 & -0.4563 \\ -0.2502 & -0.7149 & 0.5331 \\ \hline 0.4515 & -0.5542 & 0.1822 \end{array}\right]. \tag{5.51}$$

The frequency response of both the open loop and closed loop systems is given in Figure 5.4. As can be seen, the $\mathcal{H}_\infty$ norm has been reduced from 55 to 5. Note that while the $\mathcal{H}_\infty$ bounds in this case were a bit conservative, FH optimization was quite effective in reducing the $\mathcal{H}_\infty$ norm of the system.

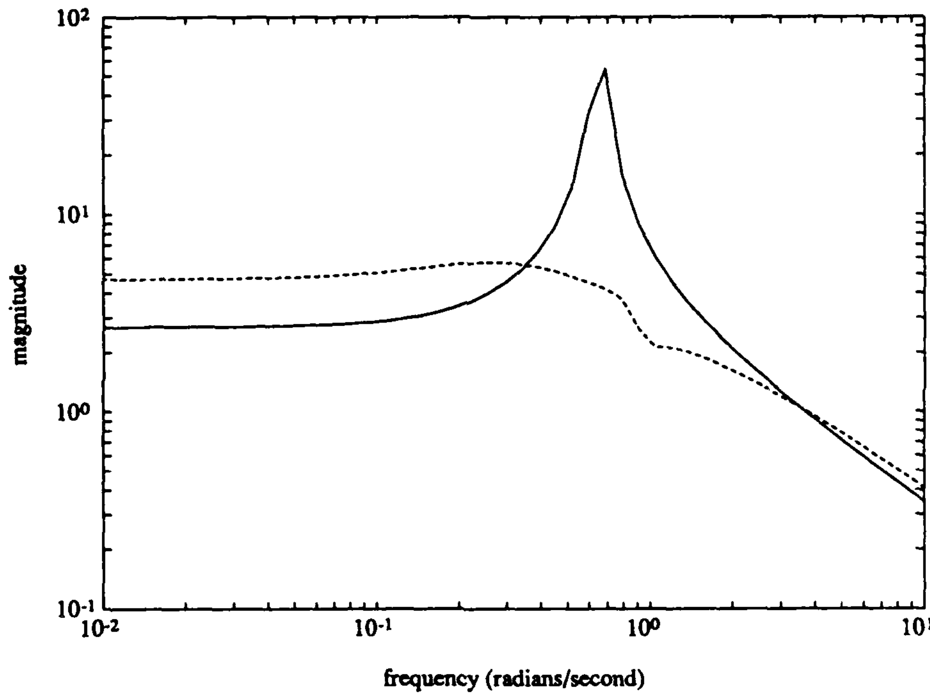


Figure 5.4: Frequency response of open loop and optimal systems.

# CHAPTER 6

# PROJECTIVE CONTROLLERS

The projective controls method [15,16] is a method for designing low order controllers for higher order systems. This approach has been studied for a variety of approaches [8,21,20,22,30,29]. The essence of the method is that it restricts the class of low order controllers considered to the subclass of controllers that retain the dominant eigenvalues and eigenstructure of a reference state feedback controlled system. Thus, given a reference system with eigenvalues given by

$$\lambda_{ref} = \lambda_{r1} \cup \lambda_{r2} \tag{6.1}$$

and the associated eigenstructure $span\{X_{r1}\}$ and $span\{X_{r1}\}$, the closed loop system under projective controls has eigenvalues

$$\lambda_{proj} = \lambda_{r1} \cup \lambda_{p2} \tag{6.2}$$

and retains the eigensubspace $span\{X_{r1}\}$, where $\lambda_{r1}$ contains the retained eigenvalues from (6.1) and $\lambda_{p2}$ contains the remaining, i.e., residual eigenvalues of the closed loop system. The family of controllers which achieve this are conveniently parameterized by a single free parameter matrix of dimension $p \times r$ where $r$ is the number of available measurements and $p$ is the order of the controller.

The retention of the eigenstructure as well as the eigenvalues of the reference system is particularily important in disturbance attenuation since the orientation of the eigenspaces influences the transfer function and system zeros. Thus, if the state feedback reference solution is determined to provide disturbance attenuation, this is reflected in the eigenstructure, and the retention of the eigenstructure together with the eigenvalues is an added incentive to the use of projective controls in disturbance attenuation.

This effect of projective controls can be characterized through the transfer function of the reference system. Let the transfer function from the disturbance input to the regulated ouput for the state feedback controller reference system be given by $G_r(s)$. This transfer function can be reduced to the series connection

$$G_r(s) = G_{r1}(s)G_{r2}(s) \tag{6.3}$$

where $G_{r1}(s)$ contains the dynamics of the reference system to be retained and $G_{r2}(s)$ contains the remaining dynamics. The corresponding transfer function under projective controls can be reduced to the series connection

$$G_p(s) = G_{r1}(s)G_{p2}(s) \tag{6.4}$$

where $G_{r1}(s)$ is called the **retained** subsystem since it contains the dynamics retained from the reference system and $G_{p2}(s)$ is called the **residual** subsystem. The order of the retained subsystem is determined by the class of controller chosen. Once the order and dynamics to be retained are chosen, the computation of the projective controller is straightforward. Three classes of controllers are considered here: static, proper and strictly proper controllers.

The reference system is determined by a state feedback controller which is chosen for its desirable properties. Many algorithms exist for designing state feedback controllers, thus the projective controls method is suitable for use in combination with many types of synthesis methods. Moreover, once the state feedback controller is determined, the projective controller is easily computed. However, stability and performance of the residual dynamics are not guaranteed.

In the case of dynamic controllers, free parameters exist which may be used to improve the performance of the residual subsystem. Section 6.5 addresses the problem of determining the free parameters.

In this chapter, expressions are shown for the projective controllers, the closed loop eigenvalues and the error between the reference system and the projective system.

## 6.1 The Reference System

A state feedback controller $u = K_o x$ applied to the plant yields the reference system

$$G_r(s) = \left[\begin{array}{c|c} F & G \\ \hline H + EK_o & 0 \end{array}\right] \tag{6.5}$$

where $F = A + BK_o$. The eigenstructure of the reference system is $FX = X\Lambda$ where $\Lambda$ is a diagonal matrix of the eigenvalues of $F$, $\lambda(F)$ and $X$ is a matrix of the associated eigenvectors.

One common design approach is LQ optimization. It has the desirable properties of producing controllers which are guaranteed to be stable through the solution of a matrix equation. In particular, the stabilizing controller which minimizes $\|G(s)\|_2$ is given by

$$u = K_2 x, \quad K_2 = -B^T M_2 \tag{6.6}$$

where $M_2 > 0$ is the solution of the algebraic Riccati equation

$$A^T M_2 + M_2 A - M_2 B B^T M_2 + H^T H = 0. \tag{6.7}$$

For details, see [12].

A stabilizing controller which guarantees $\|G(s)\|_\infty \leq \gamma$ is given by

$$u = K_\infty x, \quad K_\infty = -B^T M_\infty \tag{6.8}$$

providing there exists $M_\infty > 0$ which satisfies the algebraic Riccati equation

$$A^T M_\infty + M_\infty A - M_\infty B B^T M_\infty + \frac{1}{\gamma^2} M_\infty G G^T M_\infty + H^T H = 0. \tag{6.9}$$

For details, see [5].

## 6.2 Static Controllers

In this section, the static controller will be determined which retains the $r$ reference eigenvalues $\Lambda_r$ and associated eigenvectors $X_r$ where $r$ is the number of measured outputs.

**Theorem 13** *If $\Lambda_r$ is observable from $C$, then the unique static output feedback controller which retains the $r$ reference eigenvalues is given by*

$$C(s) = D_c \tag{6.10}$$

*where*

$$D_c = K_o N_o \tag{6.11}$$

$$N_o \triangleq X_r (C X_r)^{-1} \tag{6.12}$$

*and $X_r$ spans the eigenspace associated with the $r$ reference eigenvalues $\Lambda_r$.*

**Proof:** Since the feedback gain $D_c$ retains $[\Lambda_r, X_r]$

$$A_c X_r = (A + BD_c C)X_r = X_r \Lambda_r. \tag{6.13}$$

Also, note that

$$FX_r = (A + BK_o)X_r = X_r \Lambda_r. \tag{6.14}$$

Subtracting the two equations yields

$$BD_c CX_r = BK_o X_r. \tag{6.15}$$

Since $\Lambda_r$ is observable from $C$, $CX_r$ is invertible. Thus,

$$D_c = K_o X_r (CX_r)^{-1}. \tag{6.16}$$

■

The eigenvalues of the closed loop system consist of the retained eigenvalues and the residual eigenvalues. Note that the eigenvectors are also retained.

**Theorem 14** *The eigenvalues of the system are*

$$\lambda_c = \lambda_r \cup \lambda(A_r) \tag{6.17}$$

*where*

$$A_r \triangleq Y^T(I_n - N_o C)AY \tag{6.18}$$

*and* $Y \in \mathcal{R}^{n \times (n-r)}$ *is such that* $CY = 0$ *and* $Y^T Y = I_{n-r}$.

**Proof:** Let $T$ be given by

$$T = \begin{bmatrix} X_r & Y \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} (CX_r)^{-1}C \\ Y^T(I_n - N_o C) \end{bmatrix}. \tag{6.19}$$

Note that $U$ and $V$ exist provided that $CX_r$ is invertible which is guaranteed by the observability of $\Lambda_r$. Thus $T$ is invertible since $U$ and $V$ exist.

$$T^{-1} A_c T = \begin{bmatrix} \Lambda_r & * \\ 0 & A_r \end{bmatrix} \tag{6.20}$$

$$A_r = Y^T(I_n - N_o C)AY. \tag{6.21}$$

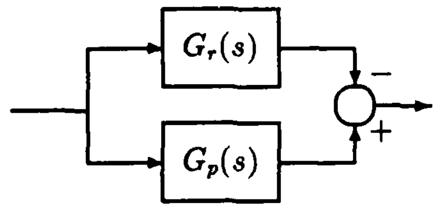


Figure 6.1: Error system.

$\blacksquare$

The following theorem forms the transfer function of the system represented by the difference between the reference and the projective systems. This system is illustrated in Figure 6.1.

**Theorem 15** *The transfer function of the error system $E(s) \triangleq G_r(s) - G_p(s)$ can be expressed as*

$$E(s) = \left[\begin{array}{c|c} F & B \\ \hline H + EK_o & E \end{array}\right] \left[\begin{array}{c|c} A_r & G_r \\ \hline K_oY & 0 \end{array}\right] \tag{6.22}$$

*where $G_r \triangleq Y^T(I_n - N_oC)G$.*

**Proof:** By definition, $E(s)$ is given by

$$E(s) = \left[\begin{array}{cc|c} F & 0 & G \\ 0 & A_c & -G \\ \hline H + EK_o & H + ED_cC & 0 \end{array}\right]. \tag{6.23}$$

Applying the state space transformation,

$$T_o \triangleq \begin{bmatrix} I_n & -X_r & -Y \\ 0 & X_r & Y \end{bmatrix}, \quad T_o^{-1} = \begin{bmatrix} I_n & I_n \\ 0 & U \\ 0 & V \end{bmatrix}. \tag{6.24}$$

$E(s)$ is given by

$$E(s) = \left[\begin{array}{ccc|c} F & 0 & -BK_oY & 0 \\ 0 & \Lambda_r & UAY & -UG \\ 0 & 0 & A_r & -VG \\ \hline H + EK_o & 0 & -EK_oY & 0 \end{array}\right]. \tag{6.25}$$

Removing the unobservable states yields

$$E(s) = \left[\begin{array}{cc|c} F & -BK_oY & 0 \\ 0 & A_r & -VG \\ \hline H+EK_o & -EK_oY & 0 \end{array}\right] = \left[\begin{array}{c|c} F & B \\ \hline H+EK_o & E \end{array}\right] \left[\begin{array}{c|c} A_r & VG \\ \hline K_oY & 0 \end{array}\right]. \tag{6.26}$$

■

## 6.3 Strictly Proper Controllers

In this section, the use of strictly proper controllers for projective controls is investigated. It is shown that the use of strictly proper controllers introduces free parameters which can be used to shape the residual dynamics.

The following theorem constructs the family of $p$-th order strictly proper controllers which retain $p$ reference eigenvalues and eigenvectors.

**Theorem 16** *The set of p-th order strictly proper controllers which retain p reference eigenvalues and eigenvectors is given by*

$$C(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] \tag{6.27}$$

*where*

$$A_c = \Lambda_p - P_oCX_p \tag{6.28}$$

$$B_c = P_o \tag{6.29}$$

$$C_c = K_oX_p, \tag{6.30}$$

*$P_o \in \mathcal{R}^{p \times r}$ is a free parameter matrix and $X_p$ spans the eigenspace associated with the p reference eigenvalues, $\Lambda_p$.*

**Proof:**

$$\begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix} \begin{bmatrix} X_p \\ W_p \end{bmatrix} = \begin{bmatrix} X_p \\ W_p \end{bmatrix} \Lambda_p \tag{6.31}$$

$$A_cW_p + B_cCX_p = W_p\Lambda_p \tag{6.32}$$

$$A_c = W_p\Lambda_pW_p^{-1} - B_cCX_pW_p^{-1} \tag{6.33}$$

$$A_c = W_p(\Lambda_p - W_p^{-1}B_cCX_p)W_p^{-1}. \tag{6.34}$$

Define $P_o \triangleq W_p^{-1}B_c$

$$A_c = W_p(\Lambda_p - P_oCX_p)W_p^{-1} \tag{6.35}$$

$$B_c = W_pP_o. \tag{6.36}$$

$$BC_cW_p + AX_p = X_p\Lambda_p \tag{6.37}$$

$$BC_cW_p + AX_p = (A + BK_o)X_p \tag{6.38}$$

$$C_cW_p = K_oX_p \tag{6.39}$$

$$C_c = K_oX_pW_p^{-1}. \tag{6.40}$$

Note that $W_p$ represents a state space transformation of the controller and thus $W_p$ is arbitrary. Choose $W_p = I_p$, then

$$A_c = \Lambda_p - P_oCX_p \tag{6.41}$$

$$B_c = P_o \tag{6.42}$$

$$C_c = K_oX_p. \tag{6.43}$$

■

**Theorem 17** *The eigenvalues of the closed loop system are*

$$\bar{\lambda}_c = \lambda_p \cup \lambda(\bar{A}_r) \tag{6.44}$$

*where*

$$\bar{A}_r \triangleq A - X_pP_oC. \tag{6.45}$$

**Proof:**

$$\bar{A}_c = \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}. \tag{6.46}$$

Let

$$\bar{T} = \begin{bmatrix} X_p & I_n \\ I_p & 0 \end{bmatrix}, \quad \bar{T}^{-1} = \begin{bmatrix} 0 & I_p \\ I_n & -X_p \end{bmatrix} \tag{6.47}$$

$$\bar{T}^{-1}\bar{A}_c\bar{T} = \begin{bmatrix} \Lambda_p & * \\ 0 & \bar{A}_r \end{bmatrix}. \tag{6.48}$$

$$\bar{A}_r = A - X_p P_o C \tag{6.49}$$

■

**Theorem 18** *The transfer function of the error system $E(s) \triangleq G_r(s) - G_p(s)$ is given by*

$$E(s) = \left[\begin{array}{c|c} F & B \\ \hline H + EK_o & E \end{array}\right] \left[\begin{array}{c|c} \bar{A}_r & G \\ \hline K_o & 0 \end{array}\right]. \tag{6.50}$$

**Proof:**

$$E(s) = \left[\begin{array}{ccc|c} F & 0 & 0 & -G \\ 0 & A & BC_c & G \\ 0 & B_cC & A_c & 0 \\ \hline H + EK_o & H & EC_c & 0 \end{array}\right]. \tag{6.51}$$

Applying the state space transformation

$$\bar{T}_o \triangleq \begin{bmatrix} I_n & -X_p & -I_n \\ 0 & X_p & I_n \\ 0 & I_p & 0 \end{bmatrix}, \quad \bar{T}_o^{-1} = \begin{bmatrix} I_n & I_n & 0 \\ 0 & 0 & I_p \\ 0 & I_n & -X_p \end{bmatrix} \tag{6.52}$$

yields

$$E(s) = \left[\begin{array}{ccc|c} F & 0 & -BK_o & 0 \\ 0 & \Lambda_p & P_oC & 0 \\ 0 & 0 & \bar{A}_r & G \\ \hline H + EK_o & 0 & -EK_o & 0 \end{array}\right]. \tag{6.53}$$

Removing the unobservable states yields

$$E(s) = \left[\begin{array}{cc|c} F & -BK_o & 0 \\ 0 & \bar{A}_r & G \\ \hline H + EK_o & -EK_o & 0 \end{array}\right] = \left[\begin{array}{c|c} F & B \\ \hline H + EK_o & E \end{array}\right] \left[\begin{array}{c|c} \bar{A}_r & G \\ \hline K_o & 0 \end{array}\right]. \tag{6.54}$$

■

## 6.4 Proper Controllers

In this section, the use of proper controllers (that is, dynamic controllers) for projective controls is investigated. It is shown that the use of proper controllers allows more of the reference dynamics to be retained and introduces free parameters which can be used to shape the residual dynamics.

The following theorem constructs the family of $p$-th order proper controllers which retain $r + p$ reference eigenvalues and eigenvectors.

**Theorem 19** *The set of all p-th order proper controllers which retain $r + p$ reference eigenvalues and eigenvectors is given by*

$$C(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \tag{6.55}$$

*where*

$$A_c = \Lambda_p + P_o C F B_o \tag{6.56}$$

$$B_c = P_o C F(N_o - B_o P_o) - \Lambda_p P_o \tag{6.57}$$

$$C_c = K_o B_o \tag{6.58}$$

$$D_c = K_o(N_o - B_o P_o), \tag{6.59}$$

*$B_o = (I - N_o C)X_p$, $P_o \in R^{p \times r}$ is a free parameter matrix and $X_r$ and $X_p$ span the reference eigenspaces associated with the reference eigenvalues, $\Lambda_r$ and $\Lambda_p$.*

**Proof:**

$$\tilde{A}_c \tilde{X}_r = \tilde{X}_r \tilde{\Lambda}_r \tag{6.60}$$

$$\begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix} \begin{bmatrix} X_p & X_r \\ W_p & W_r \end{bmatrix} = \begin{bmatrix} X_p & X_r \\ W_p & W_r \end{bmatrix} \begin{bmatrix} \Lambda_p & 0 \\ 0 & \Lambda_r \end{bmatrix} \tag{6.61}$$

$$A_c W_p + B_c C X_p = W_p \Lambda_p \tag{6.62}$$

$$A_c W_r + B_c C X_r = W_r \Lambda_r \tag{6.63}$$

$$BC_c W_p + (A + BD_cC)X_p = X_p \Lambda_p = (A + BK_o)X_p \tag{6.64}$$

$$BC_c W_r + (A + BD_cC)X_r = X_r \Lambda_r = (A + BK_o)X_r \tag{6.65}$$

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} W_p & W_r \\ CX_p & CX_r \end{bmatrix} = \begin{bmatrix} W_p\Lambda_p & W_r\Lambda_r \\ K_oX_p & K_oX_r \end{bmatrix} \tag{6.66}$$

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} W_p\Lambda_p & W_r\Lambda_r \\ K_oX_p & K_oX_r \end{bmatrix} \begin{bmatrix} W_p & W_r \\ CX_p & CX_r \end{bmatrix}^{-1}. \tag{6.67}$$

Define $L \triangleq W_p^{-1}W_r$

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} W_p\Lambda_p & W_pL\Lambda_r \\ K_oX_p & K_oX_r \end{bmatrix} \begin{bmatrix} W_p & W_pL \\ CX_p & CX_r \end{bmatrix}^{-1} \tag{6.68}$$

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} W_p & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} \Lambda_p & L\Lambda_r \\ K_oX_p & K_oX_r \end{bmatrix} \begin{bmatrix} I_p & L \\ CX_p & CX_r \end{bmatrix}^{-1} \begin{bmatrix} W_p^{-1} & 0 \\ 0 & I_r \end{bmatrix}. \tag{6.69}$$

Note that $W_p$ represents a state space transformation of the controller and thus, $W_p$ is arbitrary.

$$\begin{bmatrix} I_p & L \\ CX_p & CX_r \end{bmatrix}^{-1} = \begin{bmatrix} I_p + L\Delta^{-1}CX_p & -L\Delta^{-1} \\ -\Delta^{-1}CX_p & \Delta^{-1} \end{bmatrix} \tag{6.70}$$

$$\Delta \triangleq CX_r - CX_pL. \tag{6.71}$$

Define $P_o \triangleq L(CX_r - CX_pL)^{-1}$, then

$$\Delta^{-1} = (CX_r)^{-1}(I_r + CX_pP_o). \tag{6.72}$$

$$\begin{bmatrix} I_p & L \\ CX_p & CX_r \end{bmatrix}^{-1} = \begin{bmatrix} I_p + P_oCX_p & -P_o \\ -(CX_r)^{-1}CX_p(I_p + P_oCX_p) & (CX_r)^{-1}(I_r + CX_pP_o) \end{bmatrix} \tag{6.73}$$

Let $W_p = (I_p + P_oCX_p)$

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} (I_p + P_oCX_p)\Lambda_p & P_oCX_r\Lambda_r \\ K_oX_p & K_oX_r \end{bmatrix} \begin{bmatrix} I_p & -P_o \\ -(CX_r)^{-1}CX_p & (CX_r)^{-1}(I_r + CX_pP_o) \end{bmatrix} \tag{6.74}$$

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} \Lambda_p + P_oCF(X_p - N_oCX_p) & P_oCF(N_oCX_p - X_p)P_o + P_oCFN_o - \Lambda_pP_o \\ K_o(X_p - N_oCX_p) & K_o(N_oCX_p - X_p)P_o + K_oN_o \end{bmatrix} \tag{6.75}$$

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} \Lambda_p + P_oCFB_o & P_oCF(N_o - B_oP_o) - \Lambda_pP_o \\ K_oB_o & K_o(N_o - B_oP_o) \end{bmatrix}. \tag{6.76}$$

■

**Theorem 20** *The eigenvalues of the closed loop system are*

$$\tilde{\lambda}_c = \lambda_p \cup \lambda_r \cup \lambda(\tilde{A}_r) \tag{6.77}$$

*where*

$$\tilde{A}_r \triangleq A_r + B_o P_o C A Y \tag{6.78}$$

*and* $Y \in \mathcal{R}^{n\times(n-r)}$ *is such that* $CY = 0$ *and* $Y^T Y = I_{n-r}$.

**Proof:**

$$\tilde{A}_c = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}. \tag{6.79}$$

Let

$$\tilde{T} = \begin{bmatrix} X_p & X_r & Y \\ I_p + P_oCX_p & P_oCX_r & 0 \end{bmatrix} \tag{6.80}$$

$$\tilde{T}^{-1} = \begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix} = \begin{bmatrix} -P_oC & I_p \\ (CX_r)^{-1}C(I_n + X_pP_oC) & -(CX_r)^{-1}CX_p \\ Y^T + Y^T(B_oP_o - N_o)C & -Y^TB_o \end{bmatrix} \tag{6.81}$$

$$\tilde{T}^{-1}\tilde{A}_c\tilde{T} = \begin{bmatrix} \Lambda_p & 0 & * \\ 0 & \Lambda_r & * \\ 0 & 0 & \tilde{A}_r \end{bmatrix} \tag{6.82}$$

$$\tilde{A}_r = Y^T(I_n + (B_oP_o - N_o)C)AY = A_r + Y^TB_oP_oCAY. \tag{6.83}$$

■

**Theorem 21** *The transfer function of the error system* $E(s) \triangleq G_r(s) - G_p(s)$ *is given by*

$$E(s) = \left[\begin{array}{c|c} F & B \\ \hline H + EK_o & E \end{array}\right] \left[\begin{array}{c|c} \tilde{A}_r & G_r + Y^TB_oP_oCG \\ \hline K_oY & 0 \end{array}\right]. \tag{6.84}$$

**Proof:**

$$E(s) = \left[\begin{array}{ccc|c} F & 0 & 0 & -G \\ 0 & A + BD_cC & BC_c & G \\ 0 & B_cC & A_c & 0 \\ \hline H + EK_o & H + ED_cC & EC_c & 0 \end{array}\right]. \tag{6.85}$$

Applying the state space transformation

$$\tilde{T}_o \triangleq \begin{bmatrix} I_n & -X_p & -X_r & -Y \\ 0 & X_p & X_r & Y \\ 0 & I_p + P_o C X_p & P_o C X_r & 0 \end{bmatrix} \tag{6.86}$$

$$\tilde{T}_o^{-1} = \begin{bmatrix} I_n & I_n & 0 \\ 0 & -P_o C & I_p \\ 0 & (CX_r)^{-1}C(I_n + X_p P_o C) & -(CX_r)^{-1}CX_p \\ 0 & Y^T + Y^T(B_o P_o - N_o)C & -Y^T B_o \end{bmatrix} \tag{6.87}$$

yields

$$E(s) = \left[\begin{array}{cccc|c} F & 0 & 0 & -BK_oY & 0 \\ 0 & \Lambda_p & 0 & * & * \\ 0 & 0 & \Lambda_r & * & * \\ 0 & 0 & 0 & \tilde{A}_r & G_r + Y^T B_o P_o C G \\ \hline H + EK_o & 0 & 0 & -EK_oY & 0 \end{array}\right]. \tag{6.88}$$

Removing the unobservable states yields

$$E(s) = \left[\begin{array}{cc|c} F & -BK_oY & 0 \\ 0 & \tilde{A}_r & G_r + Y^T B_o P_o C G \\ \hline H + EK_o & -EK_oY & 0 \end{array}\right] \tag{6.89}$$

which is equivalent to

$$E(s) = \left[\begin{array}{c|c} F & B \\ \hline H + EK_o & E \end{array}\right] \left[\begin{array}{c|c} \tilde{A}_r & G_r + Y^T B_o P_o C G \\ \hline K_o Y & 0 \end{array}\right]. \tag{6.90}$$

■

## 6.5 The Auxiliary Minimization Problem

The previous sections established projective controllers in terms of free parameters. These free parameters affected the residual dynamics of the system and thus are an important part of the design problem. However, no straightforward method exists for choosing these parameters. In this section, an auxiliary minimization problem is formulated in order to determine the free parameters of the dynamic projective controllers.

The auxiliary minimization problem is as follows: Determine $P_o$ such that $\|E(s)\|_F$ is minimized.

Thus, it is natural to apply the results of Chapter 4 to determine the optimal parameters $P_o$. In the dynamic controller cases, $E(s)$ is given by

$$E(s) = E_1(s)E_2(s; P_o). \tag{6.91}$$

Note that $E(s)$ is a system of order $2n$. Thus, the auxiliary minimization problem is twice the order of the original problem. Also, the structure of $E(s)$ as shown in (6.91) cannot be exploited to simplify the problem. These problems do not prevent the FH optimization method from being solved and necessary conditions for an optimal solution can readily be found.

However, since $E_1(s)$ is independent of $P_o$, it acts only as a weighting factor and does not influence the stability or the robustness properties of the system. Thus, as a computational simplification, the auxiliary minimization problem is reduced to

$$\min_{P_o} \|E_2(s)\|_F. \tag{6.92}$$

Note that this is not the same as minimizing $E(s)$ and in general will give a different solution. However, it still represents minimizing a weighted sum of the error system $E(s)$.

In the case of strictly proper projective controllers, $E_2(s)$ is given by

$$E_2(s) = \left[\begin{array}{c|c} A - X_p P_o C & G \\ \hline K_o & 0 \end{array}\right] = \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & 0 \end{array}\right]. \tag{6.93}$$

Thus, the auxiliary minimization problem is to minimize over $P_o$,

$$J(P_o) = \operatorname{Tr} PQ, \tag{6.94}$$

subject to the two Lyapunov equations which define the controllability grammian $P$ and the observability grammian $Q$:

$$A_r P + P A_r^T + B_r B_r^T = 0 \tag{6.95}$$

$$A_r^T Q + Q A_r + C_r^T C_r = 0. \tag{6.96}$$

The necessary conditions for an optimal solution from Chapter 4 are (6.95)-(6.96) and

$$A_r^T L + L A_r + Q = 0 \tag{6.97}$$

$$A_r M + M A_r^T + P = 0 \tag{6.98}$$

$$X_p^T (LP + QM) C^T = 0 \tag{6.99}$$

where $M$ and $L$ are Lagrange multipliers for the constraints (6.95)-(6.96).

The gradient of $J$ with respect to $P_o$ can be computed for arbitrary $P_o$ as

$$\frac{dJ}{dP_o} = -2X_p^T (QL + MP) C^T \tag{6.100}$$

where $P$, $Q$, $L$ and $M$ solve the Lyapunov equations (6.95)-(6.98).

Thus, a steepest descent algorithm can be implemented by iteratively solving

$$P_o^{i+1} = P_o^i - \epsilon \left( \frac{dJ}{dP_o} \right) \tag{6.101}$$

for the optimal $P_o$.

This method of determining free parameters is employed in the design example of Chapter 8.

# CHAPTER 7

# DECENTRALIZED PROJECTIVE CONTROLS

In this chapter, projective controls method for the case of decentralized controllers is examined. Much of the work presented in this chapter is developed in [32,19,18]. As in the centralized case, the presentation here focuses on a transfer function point of view. In addition, a new formulation of the proper controller case is developed, which avoids the requirement of transforming the plant into a specific form, and results for the case of strictly proper controllers are introduced.

The main focus of this chapter is to show that the free parameters of the decentralized projective controllers can be treated in a manner similar to the centralized case. Thus, FH optimization is suitable for determining the free parameters of the controller. However, in order to efficiently apply the FH norm minimization approach to the disturbance rejection problem, a transformation developed in [32] will be applied to the system that reduces the closed loop system to a form which is linear in the free parameters.

Consider the system with $l$ feedback channels

$$P(s) = \left[\begin{array}{c|cccc} A & G & B_1 & \cdots & B_l \\ \hline H & 0 & 0 & \cdots & 0 \\ C_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_l & 0 & 0 & \cdots & 0 \end{array}\right]. \tag{7.1}$$

The goal is to determine the $l$ decentralized controllers $C_i(s)$ to achieve certain performance and robustness goals.

## 7.1 The Reference System

Given the $l$ state feedback controllers $u_i = K_{oi}x$, the reference system is given by

$$G_r(s) = \left[\begin{array}{c|c} F & G \\ \hline H & 0 \end{array}\right] \tag{7.2}$$

where

$$F = A + \sum_{i=1}^{l} B_i K_{oi}. \tag{7.3}$$

## 7.2 Static Controllers

For completeness, consider first the use of static output feedback.

**Theorem 22** *The $l$ decentralized static controller which retains $r$ reference eigenvalues is given by*

$$C_i(s) = K_i \tag{7.4}$$

*where*

$$K_i = K_{oi} N_{oi} \tag{7.5}$$

$$N_{oi} = X_r (C_i X_r)^{-1} \tag{7.6}$$

*and $X_r$ spans the eigenspace associated with the $r$ reference eigenvalues $\Lambda_r$.*

**Proof:**

$$A_c = A + B_1 K_1 C_1 + B_2 K_2 C_2. \tag{7.7}$$

$$A_c X_r = F X_r = X_r \Lambda_r. \tag{7.8}$$

■

The transfer function of the closed loop system is

$$G(s) = \left[ \begin{array}{c|c} A_c & G \\ \hline H & 0 \end{array} \right] \tag{7.9}$$

where

$$A_c = A + \sum_{i=1}^{l} B_i K_i C_i. \tag{7.10}$$

Hence, with static controls, there is no freedom available for the next design phase within the class of projective controls. This solution, however, serves as a foundation for expanding the admissible controllers by allowing dynamic controllers of given order. The advantages are twofold: more eigenvalues can be retained and free parameters are introduced to shape the residual dynamics to achieve disturbance rejection. The proper controller and strictly proper controller cases are treated seperately.

## 7.3 Strictly Proper Dynamic Controllers

**Theorem 23** *The set of all l-channel, p-th order decentralized strictly proper controllers which retain the p reference eigenvalues is given by*

$$C_i(s) = \left[\begin{array}{c|c} A_{ci} & B_{ci} \\ \hline C_{ci} & 0 \end{array}\right] \tag{7.11}$$

*where*

$$A_{ci} = \Lambda_p - P_i C_i X_p \tag{7.12}$$

$$B_{ci} = P_i \tag{7.13}$$

$$C_{ci} = K_{oi} X_p, \tag{7.14}$$

*$P_i \in R^{p\times r}$ are free parameter matrices and $X_p$ spans the reference eigenspaces associated with the p reference eigenvalues, $\Lambda_p$.*

**Proof:** ■

The transfer function of the closed loop system is then

$$\bar{G}(s) = \left[\begin{array}{c|c} \bar{A}_c & \bar{G} \\ \hline \bar{H} & 0 \end{array}\right] = \left[\begin{array}{ccc|c} A & B_1 K_{o1} X_p & B_2 K_{o2} X_p & G \\ P_1 C_1 & \Lambda_p - P_1 C_1 X_p & 0 & 0 \\ P_2 C_2 & 0 & \Lambda_p - P_2 C_2 X_p & 0 \\ \hline H & 0 & 0 & 0 \end{array}\right]. \tag{7.15}$$

Note that this system is linear in the free parameters. The freedom in the free parameters $P_1$ and $P_2$ can be used to shape the residual dynamics and, in the disturbance attenuation problem, to shape the system's transfer function. This linear representation (7.15) enables the efficient application of the FH norm approach.

## 7.4 Proper Controllers

**Theorem 24** *The set of all l-channel, p-th order, decentralized proper controllers which retain $r + p$ reference eigenvalues is given by*

$$C_i(s) = \left[\begin{array}{c|c} A_{ci} & B_{ci} \\ \hline C_{ci} & 0 \end{array}\right] \tag{7.16}$$

*where*

$$A_{ci} = \Lambda_p + P_i C_i F B_{oi} \tag{7.17}$$

$$B_{ci} = P_i C_i F(N_{oi} - B_{oi} P_i) - \Lambda_p P_i \tag{7.18}$$

$$C_{ci} = K_{oi} B_{oi} \tag{7.19}$$

$$D_{ci} = K_{oi}(N_{oi} - B_{oi} P_i), \tag{7.20}$$

$B_{oi} = (I - N_{oi} C_i) X_p$, $P_i \in R^{p \times r}$ *are free parameter matrices and* $X_r$ *and* $X_p$ *span the reference eigenspaces associated with the reference eigenvalues,* $\Lambda_r$ *and* $\Lambda_p$.

The transfer function of the closed loop system is

$$\tilde{G}(s) = \left[\begin{array}{c|c} \tilde{A}_c & \tilde{G} \\ \hline \tilde{H} & 0 \end{array}\right] = \left[\begin{array}{ccc|c} A_d & B_1 C_{c1} & B_2 C_{c2} & G \\ B_{c1} C_1 & A_{c1} & 0 & 0 \\ B_{c2} C_2 & 0 & A_{c2} & 0 \\ \hline H & 0 & 0 & 0 \end{array}\right] \tag{7.21}$$

where

$$A_d = A_c + B_1 D_{c1} C_1 + B_2 D_{c2} C_2. \tag{7.22}$$

Transforming the closed loop system by the similarity transformation

$$T = \begin{bmatrix} I_n & 0 & 0 \\ P_1 C_1 & I_p & 0 \\ P_2 C_2 & 0 & I_p \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} I_n & 0 & 0 \\ -P_1 C_1 & I_p & 0 \\ -P_2 C_2 & 0 & I_p \end{bmatrix} \tag{7.23}$$

yields

$$\hat{G}(s) = \left[\begin{array}{c|c} \hat{A}_c & \hat{G} \\ \hline \hat{H} & 0 \end{array}\right] = \left[\begin{array}{ccc|c} A_c & B_1 C_{c1} & B_2 C_{c2} & G \\ P_1 E_1 & \Lambda_p + P_1 G_{11} & -P_1 G_{12} & -P_1 C_1 G \\ P_2 E_2 & P_2 G_{21} & \Lambda_p + P_2 G_{22} & -P_2 C_2 G \\ \hline H & 0 & 0 & 0 \end{array}\right] \tag{7.24}$$

where

$$E_i = C_i(F N_{oi} C_i - A_c) \tag{7.25}$$

$$G_{ii} = C_1(F B_{o1} - B_1 C_{c1}) \tag{7.26}$$

$$G_{ij} = -C_i B_j N_{dj}, \quad i \neq j. \tag{7.27}$$

Thus, the expressions derived for $\hat{A}_c$, $\hat{G}$ and $\hat{H}$ all exhibit a linear dependence on the free parameter matrices $P_1$ and $P_2$. ($\hat{H}$ is in fact independent of the free parameter matrices.) This linear dependence can now be utilized to determine suitable $P_1$ and $P_2$ (and thus the dynamic controllers) to achieve disturbance attenuation by minimizing the FH norm.

# CHAPTER 8

# A CONTROLLER DESIGN EXAMPLE

The methods presented in the previous chapters are now applied to the problem of controlling a flexible structure. The problem explored in this design example is the improvement of the disturbance attenuation properties of a flexible structure using low order, robust control.

The structure considered here is the cruciform-shaped structure shown in Figure 8.1. Note that this is a realistic design problem since the plant model used in this design problem is taken from an actual structure and the model has been verified experimentally [10]. In addition, the structure exhibits many properties typical of flexible structures including lightly damped modes, decentralized structure and multiple controls and measurements. This structure has also been studied in [28] and [26].

The procedure used to design the controller is as follows. First, after determining that the system exhibits weak coupling between its $x$-axis and $y$-axis dynamics, the problem is separated into two parts: the design of the $x$-axis controller and the design of the $y$-axis controller. Second, a reference state feedback controller is determined which achieves the desired disturbance attenuation. Third, by selecting the proper modes to retain, the strictly proper projective controller is determined. Fourth, the free parameters in the projective controller are determined by applying FH optimization to solve the auxiliary minimization problem. Finally, the properties of the resulting closed loop system are shown.

## 8.1 Description of the Structure

The structure considered is a 45-foot lattice-type, lightweight (5 lb), flexible beam with fixed base and free tip shown in Figure 8.1 [10].

The control $u \in \mathcal{R}^2$ consists of torques applied at the base of the structure about the x and y axes. The disturbance $w \in \mathcal{R}^2$ is generated by an x-y translation applied to the base where the z-axis is taken to be the axis of the cruciform. Measurements of the system $y \in \mathcal{R}^8$ are obtained from an $x$-$y$ axis gyro and accelerometer sensors located at the tip

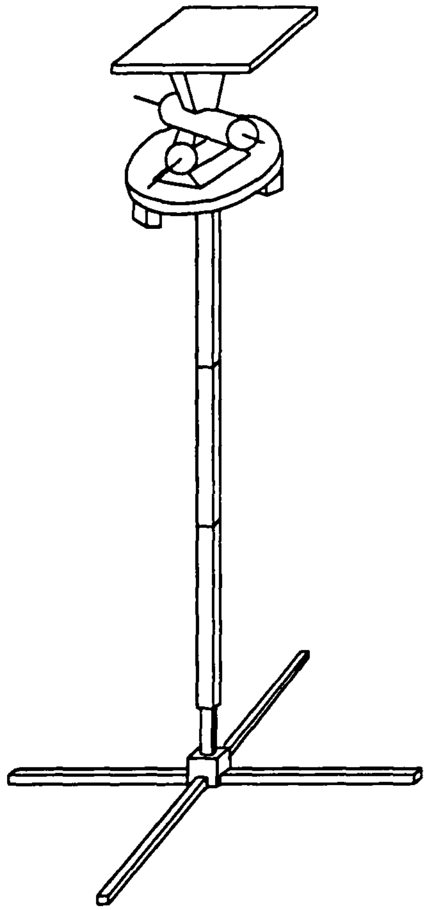

Figure 8.1: The cruciform structure.

and base of the structure. The controlled output $z \in \mathcal{R}^4$ is the position measurement at the tip and base of the structure.

The system model was obtained by truncating the lowest 20 modes of the finite element model. This results in a fortieth order state space model. The model parameters can be found in Appendix D [10].

The cruciform structure was observed to exhibit significant decoupling between the $x$ and $y$ axis dynamics. Thus, the design problem has been split into two parts: The $x$-axis dynamics and the $y$-axis dynamics. The inputs and outputs of the structure are divided as shown in Table D.1 in Appendix D. For more details see [26].

The x-axis plant model is obtained by applying a balanced model reduction on the original system using only the $x$-axis inputs and outputs. The eigenvalues of the resulting twelth order model are shown in Table 8.1 and show typical flexible structure properties,

Table 8.1: Modes of the Cruciform Model.

| Real | Imaginary | Frequency | Damping |
|---|---|---|---|
| -4.2471e-03 | ±8.4941e-01 | 8.4942e-01 | 5.0000e-03 |
| -3.4723e-02 | ±7.0339e+00 | 7.0340e+00 | 4.9365e-03 |
| -3.0800e-02 | ±7.7296e+00 | 7.7297e+00 | 3.9846e-03 |
| -1.9863e-01 | ±1.0458e+01 | 1.0460e+01 | 1.8989e-02 |
| -5.1863e-01 | ±2.5925e+01 | 2.5930e+01 | 2.0001e-02 |
| -8.2976e-01 | ±4.6092e+01 | 4.6100e+01 | 1.7999e-02 |

i.e., lightly damped and closely packed low frequency modes. The $y$-axis dynamics were treated similarly but are not shown here.

Note that model reduction was used only to remove very weakly controllable and observable modes in order to avoid neglecting modes which may be important in the design of the controller. This is possible since the projective controls method allows the use of a high order model without requiring a high order controller.

## 8.2 Design of the Controller

The goal of this section is to design a controller which improves the disturbance attenuation of the structure. The measure of disturbance attenuation used here will be the $\mathcal{H}_\infty$ norm since it represents the maximum gain of the system from disturbance to controlled output. Thus, a system such that $\|G(s)\|_\infty \leq \gamma$ is said to have disturbance attenuation $\gamma$.

From the frequency response shown in Figure 8.2, it can be seen that the open loop system has a disturbance attenuation of $\gamma$ = -20 dB. In this example, the design goal will be to improve the disturbance attenuation of the structure to $\gamma$=-40 dB using a low order, robust controller. This will be done using the projective controls method.

The first step in designing a projective controller is to find a state feedback which solves the disturbance attenuation problem. This forms the reference system for the problem. The following theorem from [26] yields the desired controller.

**Theorem 25** *Given a plant*

$$P(s) = \left[\begin{array}{c|cc} A & G & B \\ \hline H & 0 & 0 \\ I_n & 0 & 0 \end{array}\right] \tag{8.1}$$

*where there exists an $\alpha$ such that $G = B\alpha$ and a state feedback controller $C(s) = K$ where*

$$K = -R^{-1}B^T P \tag{8.2}$$

*and $P$ satisfies the Riccati equation*

$$A^T P + PA - PB^T R^{-1} BP + H^T H = 0 \tag{8.3}$$

*where $R$ satisfies*

$$\alpha^T R\alpha \leq \gamma^2, \tag{8.4}$$

*then the closed loop system satisfies the bound $\|G(s)\|_\infty \leq \gamma$.*

Making the approximation $D \approx B\alpha$ where $\alpha$=0.56, Theorem 25 can now be applied to yield the state feedback controller. The eigenvalues of the resulting closed loop system, $\lambda(F)$, are given in Table 8.2. The frequency response of the reference system is shown in Figure 8.3 and confirms that the disturbance attenuation of the reference system is $\gamma$ = -40 dB. The robustness margins are $[0, \infty)$ for gain and $\pm 90°$ for phase. Thus, this state feedback forms an acceptable reference system for the projective controls method.

First, a static projective controller is considered. Since there are $r = 4$ outputs, two complex pair modes can be retained. However, experimentation with possible retained modes does not yield any closed loop systems with the required disturbance attenuation. Since no design freedom exists for the static projective case, nothing further can be done in this case.

Since the static projective controls approach was inadequate, the strictly proper projective controls approach is now considered. Using a second order ($p = 2$) controller (higher order controllers can be considered later if this is inadequate), one complex reference mode pair can be retained. Referring to Figure 8.3, it is apparent that the lowest frequency mode of the reference system is the most important in retaining disturbance attenuation. This is the mode labeled "a" in Table 8.2. Thus, choose the retained mode to be $\lambda_p = \{a\}$.

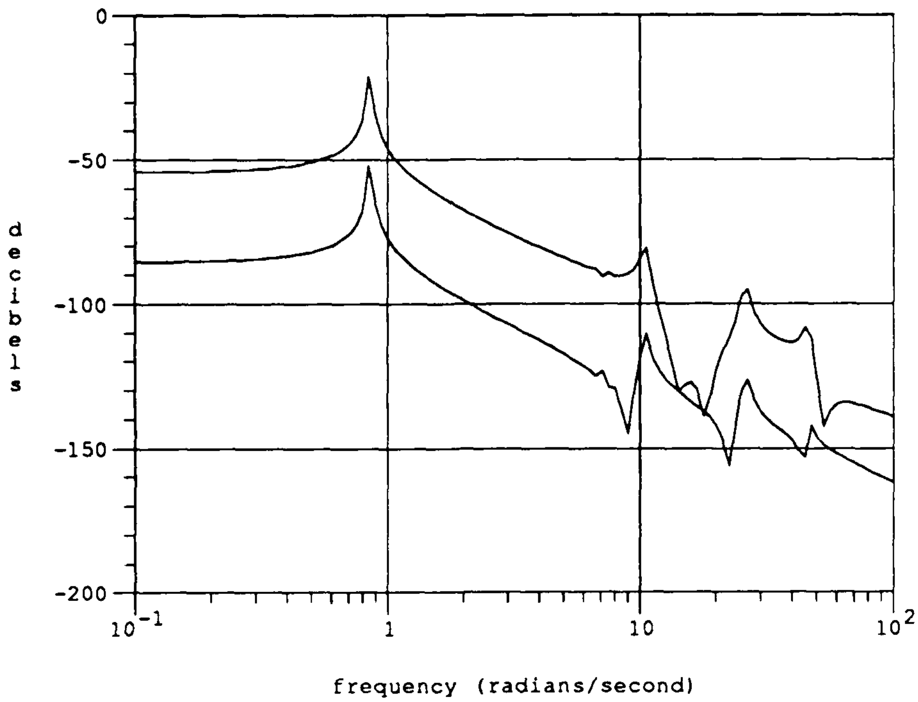


Figure 8.2: Frequency response of the open loop system.

Table 8.2: Modes of the Reference System

| | Real | Imaginary | Frequency | Damping |
|---|---|---|---|---|
| a | -1.0856e-01 | ±8.5272e-01 | 8.5960e-01 | 1.2629e-01 |
| b | -3.4723e-02 | ±7.0339e+00 | 7.0340e+00 | 4.9365e-03 |
| c | -3.0801e-02 | ±7.7296e+00 | 7.7297e+00 | 3.9848e-03 |
| d | -1.9902e-01 | ±1.0458e+01 | 1.0460e+01 | 1.9026e-02 |
| e | -5.1870e-01 | ±2.5925e+01 | 2.5930e+01 | 2.0004e-02 |
| f | -8.2977e-01 | ±4.6092e+01 | 4.6100e+01 | 1.7999e-02 |

The choice of $\lambda_p$ determines the family of strictly proper projective controllers which retain $\lambda_p$ parameterized by $P_o$. To select $P_o$, the approach of Section 6.5 is applied. Using a gradient method, the Frobenius-Hankel norm of $E_2(s)$ was minimized. The resulting controller is given in Appendix E. The frequency response of the dynamic controller associated with this choice of $P_o$ is given in Figure 8.4.

## 8.3 Evaluation of Design

The final step is to evaluate the design by examining the properties of the closed loop system. The eigenvalues of the closed loop system are shown in Table 8.3. Note that $\lambda_p = \{a\}$ has been retained. The disturbance attenuation of the full system is $\gamma$ = -40 dB as seen in Figure 8.5. Thus, the disturbance attenuation goal of the design has been met using a second order controller. The stability margins of the closed loop system are [0, 40 dB] in gain and $\pm 70°$ in phase. While these stability margins are smaller than those of the reference system, they are still quite large. However, if these margins are not satisfactory, a higher order controller may be considered.

To demonstrate the disturbance attenuation achieved by this design in the time domain, the time response of the system to a disturbance impulse is computed. The open loop response is given in Figure 8.6. Note the low damping of the low frequency mode. For the system controlled by the design given above, the response is given in Figure 8.7. In this case, the damping on the low frequency mode has increased dramatically.

A second order controller was also designed for the $y$-axis dynamics of the system in a similar manner. The resulting closed loop system with decentralized controls was seen to be stable and retain the desired disturbance attenuation properties.

Thus, the disturbance attenuation of a flexible system modeled with a fortieth-order model has been improved significantly using two second order, decentralized controllers. The use of two second order controllers is quite an improvement over the full fortieth order controller. Thus, the projective controls method with FH optimization has proven to be an effective design method in a realistic problem setting.

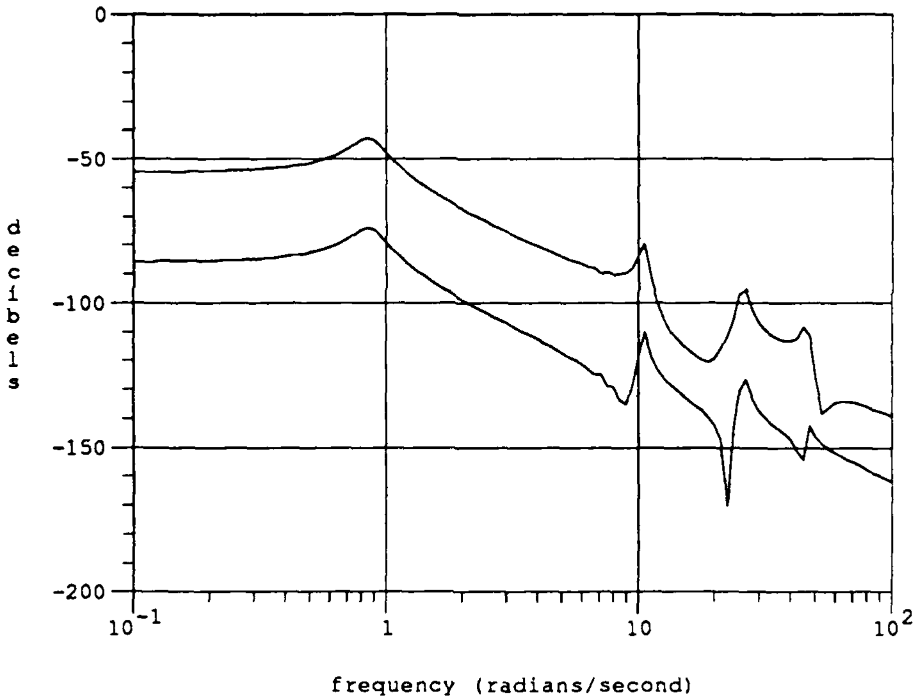


Figure 8.3: Frequency response of the reference system.

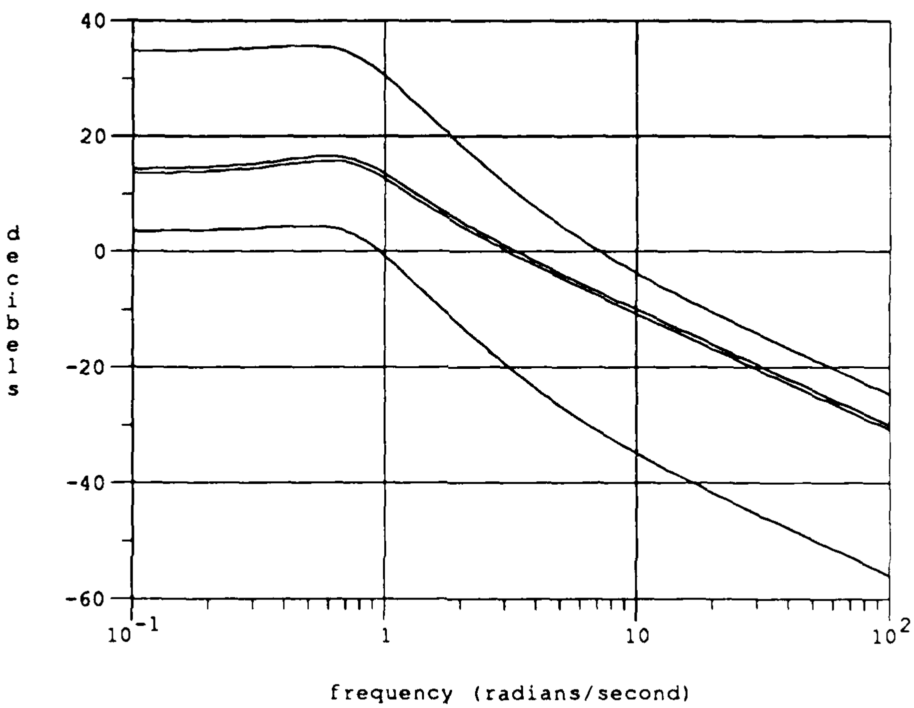


Figure 8.4: Frequency response of the controller.

Table 8.3: Modes of the Closed Loop System

| Real | Imaginary | Frequency | Damping |
|---|---|---|---|
| -2.5695e-01 | ±7.7933e-01 | 8.2059e-01 | 3.1313e-01 |
| -1.0856e-01 | ±8.5272e-01 | 8.5960e-01 | 1.2629e-01 |
| -3.4630e-02 | ±7.0333e+00 | 7.0334e+00 | 4.9237e-03 |
| -3.0564e-02 | ±7.7291e+00 | 7.7292e+00 | 3.9543e-03 |
| -1.8496e-01 | ±1.0423e+01 | 1.0425e+01 | 1.7742e-02 |
| -5.3975e-01 | ±2.5960e+01 | 2.5965e+01 | 2.0787e-02 |
| -8.1569e-01 | ±4.6082e+01 | 4.6089e+01 | 1.7698e-02 |

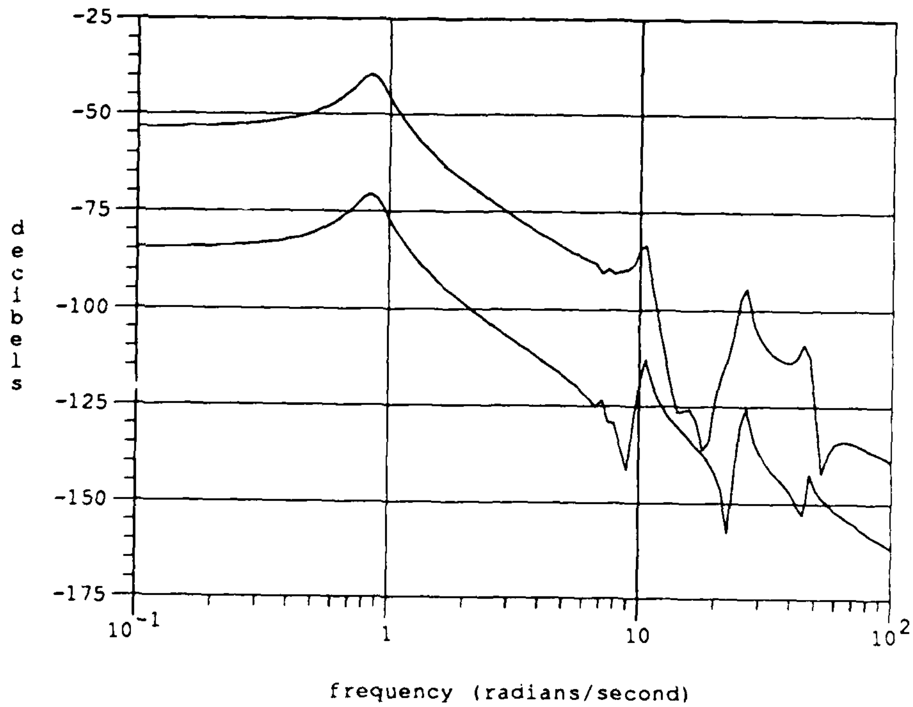


Figure 8.5: Frequency response of the closed loop system.

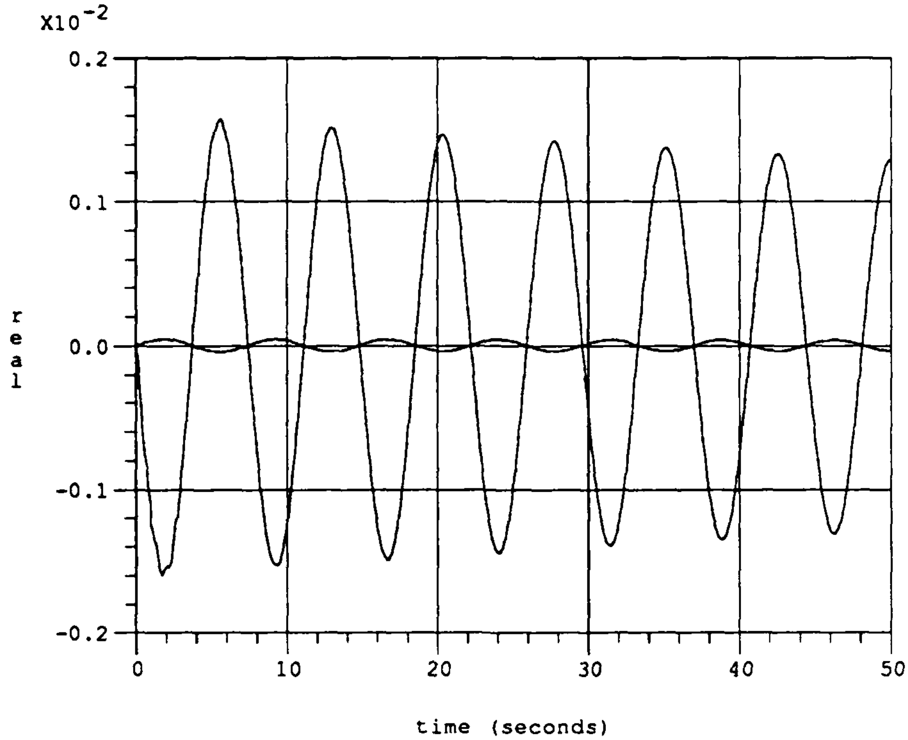


Figure 8.6: Time response of the open loop system.

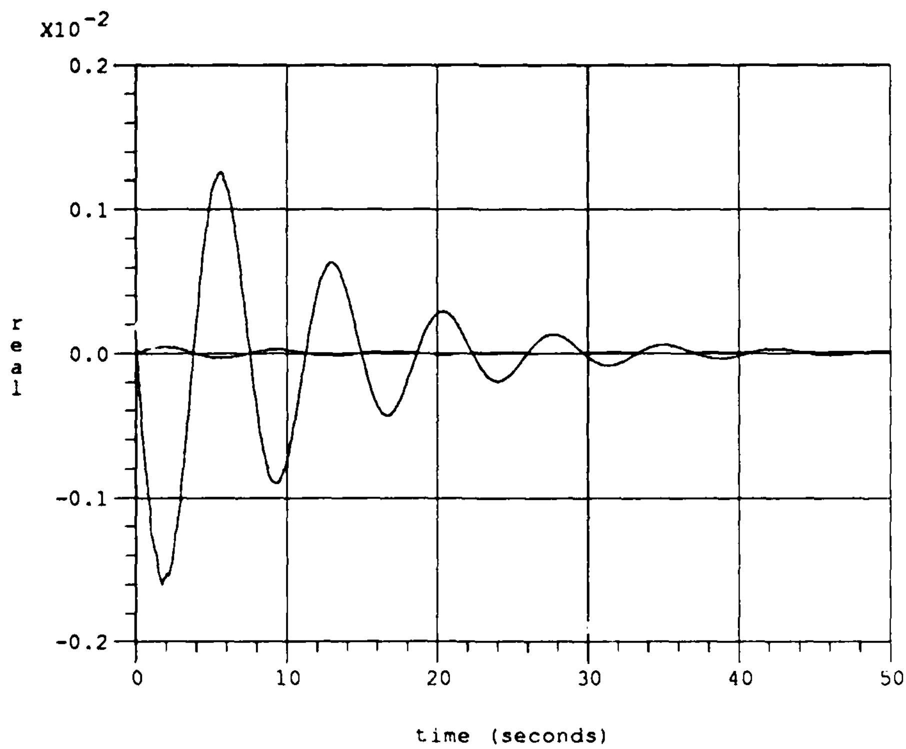


Figure 8.7: Time response of the closed loop system.

# CHAPTER 9

## CONCLUSIONS

This work has developed methods of designing low order, decentralized controllers which give robust performance in the closed loop system.

The robustness properties of the $\mathcal{H}_\infty$ norm make it an attractive choice for computing low order optimal controllers. However, no methods previously existed to solve such a problem. Necessary conditions for the solution of this problem shown in this work, however, still do not lead to effective methods for the computation of such controllers.

The FH norm, however, was shown to be a suitable norm for use in the design of low order controllers based on its ability to quantify both robustness and performance issues. In addition, it poses a much simpler computational problem which makes it suitable for application to complex problems.

The FH optimization method was used in the design of low order controllers in two distinct ways. First, necessary conditions for FH optimal controllers were derived. This led to an optimization algorithm which was demonstrated through an example. In the second approach, projective controls were used in the first phase of the design. However, the projective controllers contain free parameters which influence the residual dynamics of the closed loop system. FH optimization was then used to choose the free parameters to match the closed loop system as closely as possible with the reference system.

As an example of this approach, a design example was shown which used projective controls and FH optimization to determine the control of a flexible structure.

With the recent development of new methods to solve the full order $\mathcal{H}_\infty$ problem, the problem of determining full order, centralized controllers can be considered to be close to a solved problem. However, the situation is far different for the case of low order controllers. The problem of determining the best low order controller is still an open problem. A variety of techniques are necessary to adequately design such controllers. This work has shown some new ways to attack this problem.

# APPENDIX A

## PROPERTIES OF THE TRACE FUNCTION

**Definition 7** *Given* $A \in \mathcal{R}^{n \times n}$

$$\operatorname{Tr} A \triangleq \sum_{i=1}^{n} A_{ii} \tag{A.1}$$

In the following, assume $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, $C \in \mathcal{R}^{m \times n}$.

**Property 1**

$$\operatorname{Tr} BC = \operatorname{Tr} CB. \tag{A.2}$$

**Property 2**

$$\operatorname{Tr} A = \operatorname{Tr} A^T. \tag{A.3}$$

**Property 3**

$$\operatorname{Tr} A = \sum_{i=1}^{n} \lambda_i(A). \tag{A.4}$$

**Property 4**

$$\frac{\partial}{\partial B} \operatorname{Tr} B^T C = C. \tag{A.5}$$

# APPENDIX B

## FH OPTIMAL STATE FEEDBACK CONTROLLERS

This theorem shows that the unconstrained FH optimal controller is not equivalent to a static state feedback controller as is the case for an LQ controller.

**Theorem 26**

$$\inf_{u} G(s) < \inf_{\substack{K \\ u=Kx}} G(s). \tag{B.1}$$

**Proof:** The minimum principle is as follows: Let

$$\dot{x} = f(x,u,t), \quad x(t_o) = x_o. \tag{B.2}$$

Find the control $u$ which minimizes the criterion

$$J = \int_{t_o}^{t_f} l(x,u,t)dt + m(x(t_f)). \tag{B.3}$$

Define the Hamiltonian of the system as

$$H \equiv l + p^T f. \tag{B.4}$$

Then the necessary conditions for a minimum are

$$\begin{aligned} \dot{x} &= f(x,u,t), \quad x(t_o) = x_o \\ \dot{p} &= -\nabla_x H, \quad p(t_f) = \nabla_{x_f} m \\ 0 &= \nabla_u H. \end{aligned} \tag{B.5}$$

Now consider the LTI system,

$$\dot{x} = Ax + B_2 u, \quad x(t_o) = x_o \tag{B.6}$$

$$z = C_1 x + D_{12} w \tag{B.7}$$

$$y = C_2 x + D_{21} u. \tag{B.8}$$

For the FH norm, define the following criterion

$$J = \frac{1}{2}\int_{t_o}^{t_f} t z^T z dt. \tag{B.9}$$

Then,

$$H = \frac{1}{2}t(x^T C_1^T C_1 x + u^T u) + p^T(Ax + B_2 u). \tag{B.10}$$

This yields

$$\begin{aligned} \dot{x} &= Ax + B_2 u, \qquad & x(t_o) &= x_o \\ \dot{p} &= -tC_1^T C_1 x - A^T p, & p(t_f) &= 0 \\ 0 &= tu + B_2^T p. \end{aligned} \tag{B.11}$$

Thus, the optimal control is

$$u = -t^{-1} B_2^T p. \tag{B.12}$$

To convert the control to feedback format, assume $p(t)$ has the form,

$$p(t) = tP(t)x(t). \tag{B.13}$$

Then,

$$\dot{p} = Px + t\dot{P} + tP\dot{x}. \tag{B.14}$$

Substitution yields

$$-\dot{P} = A^T P + PA + C_1^T C_1 - PB_2 B_2^T P + \frac{1}{t}P. \tag{B.15}$$

Thus, the optimal control is given by

$$u = K(t)x \tag{B.16}$$

where

$$K(t) = -B_2^T P(t). \tag{B.17}$$

Note that as $t_o \to -\infty$, (B.15) does not have a steady-state solution. To see this, assume that a steady-state solution $\bar{P}$ exists. Then,

$$\dot{\bar{P}} = 0 \Rightarrow \frac{1}{t}\bar{P} = C \Rightarrow \bar{P} = Ct \tag{B.18}$$

which is a contradiction. Thus, a steady-state solution does not exist.

This shows that the unconstrained FH optimal controller is in fact a time-varying controller and not equivalent to a static state feedback controller. ■

# APPENDIX C

## FH OPTIMIZATION ALGORITHM

The following is a Matlab macro which implements the algorithm for determining the FH optimal controller.

```
% FH minimization
%
[nn,nz] = size(a); [nz,nm] = size(b);
[nr,nz] = size(c); [nz,nq] = size(g);
[ns,nz] = size(h); [np,nz] = size(Ac);

%Create extended system
A = [a zeros(nn,np);zeros(np,nn) zeros(np)];
B = [b zeros(nn,np); zeros(np,nm) eye(np)];
C = [c zeros(nr,np); zeros(np,nn) eye(np)];
G = [g; zeros(np,nq)];
H = [h zeros(ns,np)];
D = [d; zeros(np,nq)];
E = [e zeros(ns,np)];
K = [Dc Cc;Bc Ac];

tol = 1e-6; err = 1; ep = .1;
y1 = []; y2 = [];
while err>tol,
        Ae = A + B*K*C;
        Be = G + B*K*D;
        Ce = H + E*K*C;
        if any(real(eig(Ae))>0),
                disp('The system is unstable'),
        end

        %Calculate P and Q
        P = lyap(Ae,Be*Be');
        Q = lyap(Ae',Ce'*Ce);
        J = trace(P*Q), y1 = [y1 J];

        %Calculate L and M
        L = lyap(Ae',Q);
        M = lyap(Ae,P);

        dK = B'*(L*P+Q*M)*C' + B'*L*(G+B*K*D)*D' + E'*(H+E*K*C)*M*C';
        z = .1*norm(K)/norm(dK);
```

```
        if z < 1, ep = z;
        else, ep = 1;
        end
        K = K - ep*dK;
        err = norm(dK), y2 = [y2 err];
end
```

# APPENDIX D

# MODEL PARAMETERS

The state space model which describes the cruciform structure of Figure 8.1 is given in [10];

$$\dot{x} = Ax + Bu + Gw$$

$$y = Cx + Fu + Dw$$

$$z = Hx$$

where,

$$A = \begin{bmatrix} -2\xi\Omega & -\Omega^2 \\ I & 0 \end{bmatrix} \quad B = \begin{bmatrix} B_L \\ 0 \end{bmatrix}$$

$$G = \begin{bmatrix} D_L \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} C_r & 0 \\ -2C_a\xi\Omega & -C_a\Omega^2 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 \\ C_aB_L \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ C_aD_L \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & C_a \end{bmatrix}.$$

$$\begin{aligned} \Omega = diag[\ & .00025 \quad .00001 \quad .00083 \quad .84942 \quad .87782 \quad 2.2420 \quad 7.0290 \quad 7.2100 \quad 7.3300 \\ & 7.6228 \quad 7.7327 \quad 7.9790 \quad 8.0682 \quad 8.5011 \quad 10.460 \quad 19.910 \quad 20.380 \quad 25.930 \\ & 42.760 \quad 46.100 \ ], \end{aligned}$$

$$\begin{aligned} \xi = diag[\ & .000 \quad .000 \quad .000 \quad .005 \quad .020 \quad .005 \quad .005 \quad .005 \quad .005 \quad .005 \\ & .005 \quad .020 \quad .005 \quad .016 \quad .019 \quad .020 \quad .005 \quad .020 \quad .011 \quad .018 \ ]. \end{aligned}$$

$$
B_L = \begin{bmatrix}
7.144e-07 & 8.829e-13 \\
-6.267e-14 & -2.170e-08 \\
-1.651e-11 & 1.380e-11 \\
-1.564e-02 & 2.570e-07 \\
3.654e-07 & 1.385e-02 \\
1.043e-04 & -1.007e-04 \\
1.770e-02 & 4.871e-04 \\
-2.549e-03 & -3.550e-02 \\
-2.355e-03 & 3.588e-02 \\
1.184e-02 & -2.322e-02 \\
-1.925e-02 & -7.041e-03 \\
-5.141e-03 & -7.797e-02 \\
1.080e-02 & -2.006e-02 \\
-5.175e-04 & 1.243e-01 \\
-1.916e-01 & -2.810e-05 \\
-7.178e-05 & 1.127e-01 \\
-1.459e-03 & -5.674e-03 \\
2.716e-01 & -2.096e-06 \\
-9.120e-06 & -4.698e-02 \\
-1.770e-01 & 2.299e-06
\end{bmatrix}
\qquad
D_L = \begin{bmatrix}
1.417e-09 & 3.503e-02 \\
3.807e-02 & -6.824e-11 \\
-3.549e-07 & -1.698e-08 \\
-1.522e-07 & -6.110e-03 \\
-7.738e-03 & 1.342e-07 \\
1.393e-05 & 7.851e-06 \\
-3.123e-05 & 4.292e-04 \\
2.258e-03 & -5.985e-05 \\
-2.273e-03 & -5.48[illegible]-05 \\
1.449e-03 & 2.707e-04 \\
4.377e-04 & -4.396e-04 \\
4.806e-03 & -1.141e-04 \\
1.234e-03 & 2.386e-04 \\
-7.523e-03 & -1.114e-05 \\
1.616e-06 & -3.771e-03 \\
-5.869e-03 & -1.181e-06 \\
2.946e-04 & -2.389e-05 \\
1.052e-07 & 4.242e-03 \\
2.175e-03 & -1.293e-07 \\
-1.050e-07 & -2.473e-03
\end{bmatrix}
$$

$$
C_r^T = \begin{bmatrix}
7.144e-07 & 0.000e+00 & -3.946e-05 & 9.562e-07 & 0.000e+00 & -3.949e-05 \\
0.000e+00 & -2.170e-08 & -4.200e-05 & 0.000e+00 & -[illegible].435e-08 & -4.200e-05 \\
0.000e+00 & 0.000e+00 & 2.224e-01 & 0.000e+00 & 0.000e+00 & 2.224e-01 \\
-1.564e-02 & 2.570e-07 & -2.994e-04 & -1.710e-02 & 3.323e-07 & 2.358e-04 \\
3.654e-07 & 1.385e-02 & -3.112e-04 & 3.967e-07 & 1.778e-02 & 2.829e-04 \\
1.043e-04 & -1.007e-04 & -6.333e-02 & 5.561e-05 & -6.235e-05 & 7.231e-01 \\
1.770e-02 & 4.871e-04 & 5.976e-06 & -2.657e-02 & -4.646e-04 & -7.021e-04 \\
-2.549e-03 & -3.550e-02 & 3.897e-04 & 2.387e-03 & 3.479e-02 & -4.811e-02 \\
-2.355e-03 & 3.588e-02 & 1.220e-04 & 2.399e-03 & -3.917e-02 & -1.555e-02 \\
1.184e-02 & -2.322e-02 & 3.809e-04 & -1.353e-02 & 1.884e-02 & -5.238e-02 \\
-1.925e-02 & -7.041e-03 & 2.500e-05 & 2.752e-02 & 5.873e-03 & -3.534e-03 \\
-5.141e-03 & -7.797e-02 & -2.456e-04 & 4.962e-03 & 6.735e-02 & 3.689e-02 \\
1.080e-02 & -2.006e-02 & 3.140e-04 & -1.078e-02 & 1.835e-02 & -4.819e-02 \\
-5.175e-04 & 1.243e-01 & 3.521e-05 & 5.377e-04 & -8.600e-02 & -5.972e-03 \\
-1.916e-01 & -2.801e-05 & 3.690e-05 & 1.823e-01 & 7.747e-06 & -9.256e-03 \\
-7.178e-05 & 1.127e-01 & -1.329e-04 & -3.891e-05 & 2.829e-01 & 9.886e-02 \\
-1.459e-03 & -5.674e-03 & -2.518e-03 & -8.886e-04 & -1.464e-02 & 1.937e+00 \\
2.716e-01 & -2.096e-06 & -9.656e-06 & 2.962e-01 & -5.066e-06 & 9.887e-03 \\
-9.120e-06 & -4.698e-02 & 7.325e-06 & 1.022e-05 & 3.637e-01 & -6.133e-03 \\
-1.770e-01 & 2.299e-06 & -9.101e-06 & 3.388e-01 & -2.373e-05 & 5.256e-03
\end{bmatrix}
$$

$$
C_a^T = \begin{bmatrix}
1.417e-09 & 3.503e-02 & 3.468e-23 & 1.432e-09 & 3.502e-02 & 1.663e-17 \\
3.807e-02 & 0.000e+00 & -1.463e-24 & 3.807e-02 & 0.000e+00 & -2.187e-18 \\
-3.549e-07 & -1.698e-08 & -1.329e-26 & -3.546e-07 & -1.673e-08 & 1.440e-21 \\
-1.522e-07 & -6.110e-03 & 2.384e-15 & 4.119e-06 & 2.222e-01 & 3.055e-09 \\
-7.738e-03 & 1.342e-07 & -2.582e-15 & 2.218e-01 & -5.184e-06 & -3.310e-09 \\
1.393e-05 & 7.851e-06 & 4.611e-18 & -1.267e-03 & -1.195e-03 & 5.910e-12 \\
-3.123e-05 & 4.292e-04 & 1.387e-10 & 7.993e-05 & 1.406e-02 & 1.778e-04 \\
2.258e-03 & -5.985e-05 & -3.414e-11 & 7.045e-03 & 4.497e-03 & -4.376e-05 \\
-2.273e-03 & -5.487e-05 & 1.290e-10 & -3.180e-02 & 3.039e-03 & 1.653e-04 \\
1.449e-03 & 2.707e-04 & 9.995e-12 & -1.190e-03 & -6.003e-03 & 1.274e-05 \\
4.377e-04 & -4.396e-04 & 2.020e-10 & 1.456e-03 & -1.779e-02 & 2.588e-04 \\
4.806e-03 & -1.141e-04 & 1.740e-10 & 5.317e-02 & 5.608e-03 & 2.230e-04 \\
1.234e-03 & 2.386e-04 & 8.846e-11 & 2.096e-02 & -9.427e-03 & 1.134e-04 \\
-7.523e-03 & -1.114e-05 & 9.249e-11 & -7.424e-02 & 1.910e-04 & 1.185e-04 \\
1.616e-06 & -3.771e-03 & -1.110e-11 & 4.036e-05 & -1.311e-01 & -1.423e-05 \\
-5.869e-03 & -1.181e-06 & -7.828e-12 & 1.680e-01 & -2.111e-05 & -1.002e-05 \\
2.946e-04 & -2.389e-05 & 4.332e-13 & -9.449e-03 & -3.073e-04 & 5.543e-07 \\
1.052e-07 & 4.242e-03 & -1.050e-11 & -6.058e-06 & -1.686e-01 & -1.343e-05 \\
2.175e-03 & -1.293e-07 & -5.344e-11 & 1.906e-01 & -3.141e-06 & -6.801e-05 \\
-1.050e-07 & -2.473e-03 & -8.271e-11 & -1.476e-05 & -1.763e-01 & -1.051e-04
\end{bmatrix}
$$

The cruciform model is decentralized into $x$-axis and $y$-axis systems as shown in Table D.1 [25].

Table D.1: GHR Decentralized Model Results.

| Model | Inputs | Outputs |
|---|---|---|
| $x$-axis | $u_1, w_2$ | $z_2, z_5, y_1, y_4, y_8, y_{11}$ |
| $y$-axis | $u_2, w_1$ | $z_1, z_4, y_2, y_5, y_7, y_{10}$ |

# APPENDIX E

## CONTROLLER PARAMETERS

The decentralized second order controllers which are applied to the cruciform structure are given in the following form:

$$\left[\begin{array}{c|c} H_o & G_o \\ \hline K_o & 0 \end{array}\right].$$

The $x$-axis controller is given by

$$C_x(s) = \left[\begin{array}{cc|cccc} -5.5653e-01 & 9.6470e-01 & -8.2991e-01 & -9.0911e-01 & 3.2908e-01 & -1.1968e+01 \\ -4.3526e-01 & -2.2147e-01 & -1.1300e+00 & -1.2389e+00 & -3.1525e-01 & 1.1474e+01 \\ \hline -1.7181e+00 & -1.2874e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 \end{array}\right].$$

The $y$-axis controller is given by

$$C_y(s) = \left[\begin{array}{cc|cccc} -8.8423e-01 & 5.5382e-01 & -3.6051e-01 & -4.8280e-01 & 9.2533e-01 & -2.6680e+01 \\ -9.4732e-01 & -1.3498e-01 & 1.8310e+00 & 2.5371e+00 & 9.2186e-02 & -2.2745e+00 \\ \hline -9.7789e-02 & 1.7635e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 \end{array}\right].$$

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